## THE NOVEMBER MEETING IN CLEVELAND

The three hundred fifty-fourth meeting of the Society was held at Cleveland, Ohio, on Friday and Saturday, November 25–26, 1938, with the Case School of Applied Science and Western Reserve University as joint hosts. The attendance was about eighty, including the following fifty-four members of the Society:

G. E. Albert, W. L. Ayres, Henry Blumberg, M. G. Boyce, F. L. Brooks, O. E. Brown, R. S. Burington, W. D. Cairns, E. H. Clarke, Max Coral, Anthony Davidson, J. M. Dobbie, Ben Dushnik, T. M. Focke, L. R. Ford, Sr. M. C. Garvin, Wallace Givens, Cornelius Gouwens, V. G. Grove, Frances Harshbarger, B. A. Hausmann, T. H. Hildebrandt, C. C. Hurd, M. M. Johnson, D. K. Kazarinoff, Cornelius Lanczos, C. C. MacDuffee, H. M. MacNeille, William Marshall, E. W. Miller, Max Morris, D. C. Morrow, J. R. Musselman, A. L. Nelson, C. J. Nesbitt, Rufus Oldenburger, L. F. Ollmann, E. J. Olson, E. W. Paxson, I. E. Perlin, R. M. Peters, Everett Pitcher, E. D. Rainville, R. F. Rinehart, E. H. Rothe, George Sauté, W. G. Simon, E. C. Stopher, C. F. Thomas, R. M. Thrall, E. P. Vance, G. B. Van Schaack, C. O. Williamson, C. H. Yeaton.

The meeting opened Friday afternoon with a session devoted to short papers, followed by an address by Professor V. G. Grove of Michigan State College entitled *A tensor analysis for a $V_k$ in a projective space $S_n$*. On Saturday morning Professor C. C. MacDuffee of the University of Wisconsin delivered an address entitled *Modules in algebraic fields*. This was followed by another session of contributed papers.

On Friday evening a dinner was held in Haydn Hall, which was attended by fifty-eight persons. Professor R. S. Burington presided at the dinner and President Winfred G. Leutner of Western Reserve University and President William Wickenden of the Case School of Applied Science welcomed the members of the Society on behalf of the two host institutions. Professor W. D. Cairns then spoke on the early history of the Mathematical Association of America, its relations to the Society, and some interesting experiences as Secretary of the Association.

Dean W. G. Simon presided at the Friday afternoon session and Professor L. R. Ford presided at the session on Saturday morning. At the Saturday morning session Professor A. L. Nelson presented a resolution of appreciation to the Departments of Mathematics of the Case School of Applied Science and Western Reserve University for their many courtesies extended during the meeting. The titles of papers read at the meeting follow. Those whose abstract numbers

are followed by the letter *t*, were read by title. Mr. J. F. Kenney was introduced by Professor D. R. Curtiss. Papers 1–8 were presented Friday afternoon and papers 9–15 on Saturday morning.

1. Max Coral: *On fields for double integral variation problems.* (Abstract 44-11-442.)

2. Everett Pitcher: *A simpler proof of a theorem of Morse.* (Abstract 44-11-454.)

3. C. J. Nesbitt: *Note on primary algebras.* (Abstract 44-11-452.)

4. Rufus Oldenburger: *Higher-dimensional determinants.* (Abstract 44-11-453.)

5. Wallace Givens: *Signatures of Lorentz matrices.* (Abstract 44-11-446.)

6. R. S. Burington: *On the congruence of matrices and associated circavariant matrices.* (Abstract 44-11-441.)

7. D. K. Kazarinoff: *Remark concerning Desargues' triangle theorem.* (Abstract 44-11-447.)

8. J. R. Musselman: *The equation of motion of equal maps.* (Abstract 44-11-451.)

9. O. E. Brown: *A least squares machine.* Preliminary report. (Abstract 44-11-440.)

10. Cornelius Lanczos: *A simple interpolation method for the representation of rugged curves.* (Abstract 44-11-449.)

11. J. M. Dobbie: *A note on a convergence proof for Fourier series.* (Abstract 44-11-445.)

12. G. E. Albert: *Asymptotic forms for a general class of hypergeometric functions with applications to the generalized Legendre functions.* (Abstract 44-9-386.)

13. E. D. Rainville: *Linear differential invariants related to the Laplace integral transformation.* (Abstract 44-11-455.)

14. E. H. Rothe: *Asymptotic solution of a boundary value problem.* (Abstract 44-11-456.)

15. R. F. Rinehart: *An interpretation of the inertia of the discriminant matrices of an associative algebra.* (Abstract 45-1-57.)

16. Reinhold Baer: *Nets and groups.* (Abstract 44-11-438-*t*.)

17. J. F. Kenney: *The regression systems of two sums having random elements in common.* (Abstract 44-11-448-*t*.)

18. M. A. Basoco: *On certain arithmetical functions due to G. Humbert.* (Abstract 44-11-439-*t*.)

19. H. V. Craig: *On extensors and a euclidean basis for higher order spaces.* (Abstract 44-11-443-*t*.)

20. Saunders MacLane: *Steinitz field towers for modular fields.* (Abstract 44-11-450-*t*.)

21. Reinhold Baer: *Groups with abelian norm quotient group.* (Abstract 44-11-437-*t*.)

22. Reinhold Baer: *Almost Hamiltonian groups.* (Abstract 44-11-436-*t*.)

23. J. J. DeCicco: *The analogue of the nine-point circle in the associated Kasner plane.* (Abstract 44-11-444-*t*.)

W. L. AYRES,
*Associate Secretary*

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## THE NOVEMBER MEETING IN LOS ANGELES

The three hundred fifty-fifth meeting of the American Mathematical Society was held at the University of California at Los Angeles on Saturday, November 26, 1938. The attendance included the following thirty-eight members of the Society:

L. J. Adams, O. W. Albert, L. D. Ames, H. A. Arnold, H. M. Bacon, Harry Bateman, Clifford Bell, E. T. Bell, Myrtie Collier, P. H. Daus, R. P. Dilworth, C. H. Dix, H. E. Glazier, H. J. Hamilton, E. R. Hedrick, D. G. Humm, G. H. Hunt, Glenn James, R. D. James, A. T. Lonseth, W. A. Manning, W. E. Mason, A. D. Michal, R. M. Robinson, A. A. Shaw, G. E. F. Sherwood, D. V. Steed, Gabor Szegö, A. E. Taylor, T. Y. Thomas, S. E. Urner, J. V. Uspensky, F. A. Valentine, D. E. Whelan, P. A. White, W. M. Whyburn, E. R. Worthington, Max Zorn.

The meeting was called to order at 10:00 A.M., and was presided over at different times by Professors E. T. Bell, R. D. James, Gabor Szegö, and T. Y. Thomas.

On Saturday afternoon a tea for members of the Society and their friends was given by Professor and Mrs. E. R. Hedrick at their home.

Titles of papers read at the meeting follow. Those whose abstract numbers are followed by the letter *t* were read by title.

1. R. M. Robinson: *A generalization of Picard's and related theorems.* (Abstract 44-9-405.)

2. A. A. Shaw: *Mathematical passages in the works of David the Invincible.* (Abstract 44-11-428.)

3. F. A. Valentine: *The problem of Lagrange with finite inequalities as side conditions.* (Abstract 44-11-433.)

4. H. J. Hamilton: *Preservation of partial limits in multiple sequence transformations.* (Abstract 44-11-422.)

5. A. E. Taylor: *Reflexive Banach spaces.* I. Preliminary report. (Abstract 44-11-430.)

6. R. D. James: *Note on integers which are not sums of three squares.* (Abstract 44-11-425.)

7. R. P. Dilworth: *Archimedean residuated lattices.* (Abstract 44-11-420.)

8. H. A. Arnold: *Topological order in Kantorovitch spaces.* (Abstract 44-11-418.)

9. A. D. Michal: *General Riemannian differential geometry with abstract coordinates and intrinsic inner product.* (Abstract 44-11-426-*t*.)

10. A. D. Michal and D. H. Hyers: *Normal coordinates in general differential geometries with inter-space inner product.* (Abstract 44-11-427-*t*.)

11. Max Wyman: *The simultaneous theory of a linear connection, and a non-holonomic linear connection in a general geometry with Banach coordinates.* (Abstract 44-11-434.)

12. D. W. Hall (National Research Fellow): *On pointwise periodic homeomorphisms.* (Abstract 44-11-421-*t*.)

13. A. E. Taylor: *Reflexive Banach spaces.* II. Preliminary report. (Abstract 44-11-431-*t*.)

14. A. E. Taylor: *Banach spaces whose unit sphere is weakly compact.* (Abstract 44-11-432-*t*.)

15. Clifford Bell: *Plane curves with pseudo rhamphoid cusps.* (Abstract 44-11-419-*t*.)

16. A. A. Shaw: *An astronomical passage of antiquity.* (Abstract 44-11-429-*t*.)

17. O. G. Harrold: *On hereditary arc sums.* (Abstract 44-11-423-*t*.)

18. O. G. Harrold: *Concerning 2–1 transformations on an arc.* Preliminary report. (Abstract 44-11-424-*t*.)

19. Edmund Pinney: *General geodesic coordinates of order r.* Preliminary report. (Abstract 45-1-52.)

T. M. PUTNAM,
*Associate Secretary*

## THE ANNUAL MEETING OF THE SOCIETY

The forty-fifth Annual Meeting of the American Mathematical Society was held at Richmond and Williamsburg, Virginia, from Tuesday to Friday, December 27–30, 1938, in conjunction with the meetings of the American Association for the Advancement of Science and the Mathematical Association of America.

There were three general and seven sectional sessions of the Society, in which four addresses and ninety-five research papers (fifty-four in person) were given. Sessions on Wednesday morning were held in the Broadcasting Studio of the Mosque, Richmond. All other sessions were held in Washington Hall and Rogers Hall at the College of William and Mary, Williamsburg.

Arrangements for the meetings were made by a committee consisting of Professors J. M. Stetson (chairman), E. J. McShane, G. A. Larew, and C. H. Wheeler. In Richmond the headquarters were at the Hotel William Byrd. In Williamsburg, the College of William and Mary generously made dormitory rooms available for the use of the mathematicians.

The attendance of approximately three hundred included the following two hundred thirty members of the Society:

C. R. Adams, R. B. Adams, V. W. Adkisson, R. P. Agnew, C. B. Allendoerfer, R. C. Archibald, C. S. Atchison, N. H. Ball, M. F. Becker, A. A. Bennett, G. D. Birkhoff, L. M. Blumenthal, Paul Boeder, H. F. Bohnenblust, H. E. Bray, W. C. Brenke, J. L. Brenner, F. L. Brooks, E. T. Browne, Herbert Busemann, J. H. Bushey, Jewell H. Bushey, W. E. Byrne, W. D. Cairns, J. W. Calkin, H. H. Campaigne, E. W. Cannon, M. E. Carlen, C. L. Carroll, I. S. Carroll, H. C. Carter, W. B. Carver, J. W. Cell, Claude Chevalley, Alonzo Church, J. M. Clarkson, Abraham Cohen, L. W. Cohen, Nancy Cole, J. B. Coleman, J. L. Coolidge, W. A. Cordrey, Richard Courant, E. B. Cowley, Sr. J. S. Creane, H. B. Curry, D. R. Curtiss, E. H. Cutler, M. D. Darkow, H. T. Davis, F. F. Decker, L. L. Dines, Arnold Dresden, F. G. Dressel, R. J. Duffin, W. L. Duren, L. A. Dye, Paul Erdös, W. H. Erskine, E. J. Finan, D. A. Flanders, L. R. Ford, Tomlinson Fort, R. H. Fox, J. S. Frame, K. O. Friedrichs, T. C. Fry, R. E. Gaines, J. J. Gergen, F. J. Gerst, Wallace Givens, Kurt Gödel, M. C. Graustein, W. C. Graustein, C. H. Graves, D. W. Hall, Israel Halperin, Harris Hancock, M. L. Hartung, N. A. Haynes, O. C. Hazlett, L. B. Hedge, E. R. Hedrick, E. H. C. Hildebrandt, Einar Hille, D. L. Holl, T. R. Hollcroft, M. A. Holly, Charles Hopkins, V. A. Hoyle, W. A. Hurwitz, Dunham Jackson, Nathan Jacobson, C. G. Jaeger, Walter Jennings, Fritz John, F. E. Johnston, B. W. Jones, F. B. Jones, H. A. Jordan, J. L. Kelley, A. J. Kempner, S. C. Kleene, J. R. Kline, R. L. Korgen, H. L. Krall, W. D. Lambert, F. L. Lamoreau, O. E. Lancaster, Lincoln LaPaz, G. A. Larew, C. G. Latimer, V. S. Lawrence, B. A. Lengyel, D. C. Lewis, J. B. Linker, S. B. Littauer, M. I. Logsdon, R. G. Lubben, N. H. McCoy, E. A. McDougle, E. J. McShane, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, H. F. MacNeish, Dorothy Manning, W. A. Manning, A. J. Maria, William Marshall, M. H. Martin,

A. K. Mitchell, Virginia Modesitt, E. C. Molina, C. N. Moore, R. L. Moore, T. W. Moore, E. M. Morenus, W. K. Morrill, E. J. Moulton, C. G. Mumford, F. L. Munroe, C. W. Munshower, F. D. Murnaghan, S. B. Myers, John von Neumann, C. O. Oakley, G. A. O'Donnell, F. W. Owens, H. B. Owens, J. C. Oxtoby, E. K. Paxton, A. M. Pegram, E. W. Pehrson, H. A. Perkins, B. J. Pettis, C. S. Pettis, Everett Pitcher, M. H. Plass, G. B. Price, W. T. Puckett, H. A. Rademacher, J. F. Randolph, Herbert ReBarker, M. S. Rees, W. D. Reeve, R. G. D. Richardson, R. F. Rinehart, E. K. Ritter, H. E. Robbins, J. H. Roberts, H. A. Robinson, Barkley Rosser, H. G. Russell, S. T. Sanders, O. F. G. Schilling, I. J. Schoenberg, G. E. Schweigert, W. T. Scott, Wladimir Seidel, C. G. Shover, R. G. Simond, Abraham Sinkov, F. C. Smith, J. P. Smith, R. G. Smith, W. F. Smith, Virgil Snyder, F. W. Sohon, C. A. Spicer, M. E. Stark, N. E. Steenrod, J. M. Stetson, C. N. Stokes, M. H. Stone, R. R. Struik, Alvin Sugar, Otto Szász, J. D. Tamarkin, J. H. Taylor, M. E. Taylor, J. M. Thomas, E. W. Titt, C. B. Tompkins, C. C. Torrance, E. M. Torrance, W. J. Trjitzinsky, B. M. Turner, S. M. Ulam, A. D. Wallace, S. E. Warschawski, J. H. Weaver, Warren Weaver, J. V. Wehausen, M. J. Weiss, M. E. Wells, V. H. Wells, A. P. Wheeler, C. H. Wheeler, G. T. Whyburn, D. V. Widder, Norbert Wiener, C. W. Williams, K. P. Williams, E. W. Wilson, H. A. Wood, F. L. Wren, B. F. Yanney, M. M. Young, H. S. Zuckerman.

On Tuesday evening, Dean G. D. Birkhoff delivered an address as retiring President of the A.A.A.S. on the subject *Intuition, reason and faith in science.* The address was followed by a reception, for all members of the Association and affiliated societies, at the Hotel Jefferson.

On Wednesday morning, at 9:30 A.M., at a joint session of the Society, the Mathematical Association of America, and Sections A and E of the A.A.A.S., Professor W. D. Cairns gave an address as retiring Vice President of the A.A.A.S. and Chairman of Section A. The subject was *Seismology from a mathematical viewpoint.* Professor J. R. Kline, Chairman of Section A, presided at this session.

On Wednesday morning, at 10:45 A.M., at a joint session of the Society, the Mathematical Association of America, and Sections A and L of the A.A.A.S., Professor R. C. Archibald gave an address as retiring Vice President of the A.A.A.S. and Chairman of Section L. His subject was *Mathematicians, and poetry and drama.* Professor D. R. Curtiss presided.

Wednesday afternoon there were two sectional sessions of the Society, one for Analysis and one for Topology and Geometry. Professors M. H. Stone and G. T. Whyburn presided. Thursday morning there were three sessions, for Analysis, Calculus of Variations and Applied Mathematics, and Algebra, at which Professor J. J. Gergen, Dr. Warren Weaver, and Professor N. H. McCoy presided. There were also two sessions on Friday morning, one for Analysis and one for Topology, at which Professors H. E. Bray and R. L. Moore presided.

The general session Thursday afternoon began with the business meeting and annual election. Following this, Professor R. L. Moore gave his address as the retiring President of the Society. His subject was *On certain abstract spaces*. Professor H. A. Rademacher then gave an address, by invitation of the Program Committee, on *Fourier expansions of modular functions and theorems on partitions*. At the close of the general session, the Bôcher Memorial Prize was awarded to Professor John von Neumann, who gave a résumé of the paper for which the prize was awarded (*Almost periodic functions and groups*, Transactions of this Society, vol. 36 (1934), pp. 445–492). Vice President C. R. Adams presided at this session.

Sessions of the Mathematical Association of America were held on Friday afternoon and Saturday morning and of the National Council of Teachers of Mathematics, Friday morning and afternoon.

A joint dinner for members of the mathematical organizations and their guests was held on Thursday evening in the Dining Hall, College of William and Mary. The attendance was three hundred and seven. Professor F. D. Murnaghan was toastmaster. There were six speakers: President J. S. Bryan of the College of William and Mary, an address of welcome to Williamsburg; Professor G. T. Whyburn, representing mathematicians of the South; Professor J. L. Coolidge, on the history of mathematics in Virginia; Professor R. L. Moore, retiring President of the Society, representing the Society; Professor W. B. Carver, President-elect of the Association, representing the Association; Dean R. G. D. Richardson, Secretary of the Society, on the proposition of starting an abstract journal in America. Professor L. R. Ford presented resolutions, which were adopted unanimously, thanking the College of William and Mary, the local committee, and all those who assisted them for their excellent arrangements and cordial hospitality.

There were excursions to historic places in Richmond Tuesday afternoon and Wednesday morning, and also to Jamestown and Yorktown Thursday afternoon. Members and guests visited the Williamsburg Restoration Buildings at various times during the meeting.

At the meeting of the Board of Trustees at 6:30 P.M., Tuesday, December 27, in the Hotel William Byrd, there was not a quorum present. An adjourned meeting, therefore, was held on December 31, at 6:30 P.M., in the Men's Faculty Club of Columbia University, New York City.

The Council met on Thursday, December 29, at 12:00 M. in the Dining Hall of the College of William and Mary, and at 10:30 P.M. in Room 200 of Washington Hall.

The Secretary announced the election of the following twenty-one persons to membership in the Society:

Professor Sam Kelley Bright, Blinn College, Brenham, Texas;
Mr. Richard Hubert Bruck, University of Toronto;
Professor Alexander V. Bushkovitch, College of Charleston, Charleston, S. C.;
Mr. John Chesna, Bausch and Lomb Optical Company, Rochester, N. Y.;
Mr. Wilfred Joseph Dixon, University of Wisconsin;
Professor William Cletus Doyle, Rockhurst College, Kansas City, Mo.;
Professor William Louis Fields, Agricultural, Mechanical, and Normal College, Pine Bluff, Ark.;
Sister M. Immaculata Garrahan, College Misericordia, Dallas, Pa.;
Mr. Robert Ewing Greenwood, Jr., University of Texas;
Dr. Olaf Helmer, University of Illinois;
Mr. Robert B. Kleinschmidt, Monmouth Junior College, Long Branch, N. J.;
Mr. Kenneth Floyd McLaughlin, Des Moines, Iowa;
Mr. Don Dalzell Miller, Ohio University;
Professor Jerzy Neyman, University of California;
Mr. Sidney Thomas Parker, Hobart College;
Professor Glenn Harold Peebles, University of Minnesota;
Mr. Clement Russell Phelps, Georgia School of Technology;
Mr. William Haddock Simons, University of California;
Professor John P. Smith, S.J., St. Peter's College, Jersey City, N. J.;
Professor Kadri Hafiz Tukan, Najah National College, Nablus, Palestine;
Dr. Henry Allen Wood, Connecticut State College.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Acadia University: Dr. Moray St. John Macphail.
University of California at Los Angeles: Messrs. Paul Joseph Kelly and Burris Palmer Taylor.
University of Chicago: Messrs. Morris Bloom, Ralph Nathanael Johanson, Albert Neuhaus, and George W. Whitehead.
College of the City of New York: Mr. Herbert L. Mintzer.
Cornell University: Dr. Samuel Saslaw and Mr. Gordon Loftis Walker.
University of Illinois: Messrs. Paul Hamilton Anderson, Philip Wilkinson Carruth, Benjamin Epstein, James Francis Heyda, Franz Edward Hohn, Thomas Eddy Judd, Paul Edison Lewis, Kaj Leo Nielsen, Charles Frederick Strobel, and Chester William Topp.
Institute for Advanced Study: Dr. Valentin Bargmann, Professor Claude Chevalley (University of Rennes, Rennes, France), Dr. Paul Erdös, Dr. Herbert Ellis Robbins, and Mr. Jacobus Stephanus de Wet.
The State University of Iowa: Mr. John Wesley Kitchens.
The Johns Hopkins University: Mr. Abraham Seidenberg.
University of Kentucky: Mr. James Malcolm Boswell.
Massachusetts Institute of Technology: Messrs. Abe Gelbart, Abraham Schwartz, and Samuel Woodard Stewart.
University of Michigan: Dr. Preston Clarence Hammer, Messrs. Samuel Kaplan, Horace Komm, John William Odle, and Charles E. Rickart.
Michigan State College of Agriculture and Applied Science: Mr. John William Zimmer.

Northwestern University: Mr. Paul B. Burcham, Miss Mary Margaret Handel, Messrs. Joseph Findlay Paydon and Richard Harlan Stark.
Syracuse University: Mr. John Emerson Hart.
University of Virginia: Mr. Richard Cyrus Morrow.
University of Washington: Messrs. Bradford Henry Arnold and George Arthur Whetstone.
Wayne University: Miss Margaret Jean Dunford.
Wesleyan University: Mr. James Henry Perryman.

The following appointments by President R. L. Moore were reported: as representative of the Society at the inauguration of Charles Burgess Ketcham as President of Mount Union College, Alliance, Ohio, on October 20, Professor B. F. Yanney; as representative of the Society at the inauguration of Karl Clayton Leebrick as President of Kent State University of Ohio, Kent, Ohio, on October 21, Dean W. G. Simon; as representative of the Society at the inauguration of Leonard Carmichael as President of Tufts College, Medford, Massachusetts, on November 4, Professor E. V. Huntington; as representative of the Society at the installation of Edwin Sharp Burdell as Director of The Cooper Union for the Advancement of Science and Art, New York, N. Y., on November 3, Professor W. B. Fite; as tellers for the election at the Annual Meeting, Professor C. H. Wheeler and Dr. E. R. Lorch; as auditors of the Society's books for the year 1938, Professors R. G. Archibald and A. E. Meder, Jr.; as a committee to nominate a representative on the National Research Council for a period of three years beginning July, 1939, and two representatives on the Council of the American Association for the Advancement of Science for the year 1939, Dean G. D. Birkhoff (chairman), Professor G. A. Bliss, and Dean L. P. Eisenhart; as committees to select hour speakers for various meetings: as the Committee for Annual and Summer Meetings, the Secretary (chairman), Professors A. B. Coble (one year) and J. R. Kline (two years); as the Committee for Eastern Sectional Meetings, the secretary for those meetings (chairman), Dean L. P. Eisenhart (two years), and Professor J. L. Walsh (one year); as the Committee for Western Sectional Meetings, the secretary for those meetings (chairman), Professors G. A. Bliss (two years) and T. H. Hildebrandt (one year); as the Committee for Far Western Sectional Meetings, the secretary for those meetings (chairman), Provost E. R. Hedrick (two years), and Professor G. C. Evans (one year).

The Secretary reported that the ordinary membership in the Society is now 2,200, including 151 nominees of institutional members and 78 life members. There are also 86 institutional members. The total attendance of members at all meetings in 1938 was 1,435; the

number of papers read was 475; the number of invited addresses was 21; the number of members attending at least one meeting was 848.

At the annual election which closed on December 29, and at which 373 votes were cast (349 for the list nominated by the Council), the following officers were elected:

*President*, Professor G. C. Evans.
*Vice President*, Professor C. R. Adams.
*Associate Secretary*, Professor T. R. Hollcroft.
*Treasurer*, Professor B. P. Gill.
*Librarian*, Professor R. C. Archibald.
*Members of the Editorial Committee of the Bulletin*, Professors A. A. Albert and Tomlinson Fort.
*Member of the Editorial Committee of the Transactions*, Professor W. C. Graustein.
*Member of the Editorial Committee of the Colloquium Publications*, Professor M. H. Stone.
*Member of the Editorial Board of the American Journal*, Professor R. L. Wilder.
*Members of the Council*, Professors H. E. Bray, Alonzo Church, E. J. McShane, H. P. Robertson, and Hermann Weyl.
*Members of the Board of Trustees*, Professor W. B. Fite, Dr. Robert Henderson, Professors W. R. Longley and G. W. Mullins, and Dean R. G. D. Richardson.

The reports of the Treasurer and auditors (Professors R. G. Archibald and A. E. Meder, Jr.) showed a balance of $12,635.94 exclusive of the balances in the Colloquium, Sinking Fund, and special funds. The Society's investments including Endowment Fund, Sinking Fund, and other special funds had a market value on November 30, 1938, of approximately $90,000. The net interest income for the year was $3,762.48; dues from Institutional Memberships amounted to $6,455; dues from Contributing Memberships, $1,124.96; and dues from Ordinary Memberships were $13,406.16. The Colloquium Fund showed a balance of $10,813.91. The total received from the sale of the Society's publications was $10,421.64. The Board of Trustees adopted a budget for 1939 showing estimated expenditures and receipts of the Society (exclusive of Congress funds) as $36,250 and $32,550, respectively.

The Librarian reported that the Library of the Society now contains about 9,150 volumes.

Invitations were received for the Annual Meeting of 1940 from Louisiana State University, University of Pennsylvania, and Lehigh

University. A committee was appointed to make recommendations regarding the choice of place.

An invitation was received from Northwestern University for the Society to meet at that institution in the summer of 1943, this being the fiftieth anniversary of the first Colloquium of the Society, when Professor Felix Klein spoke at Evanston.

Times and places of meetings in 1939 were fixed as follows: Stanford University, April 15; Madison, September 5–8; New York City, October 28.

Dr. Thornton C. Fry, executive secretary of the Committee on the Semicentennial Celebration, made a final report. He stated that of the $5,500 set aside for publication expenses, there remains a balance of $854.75 and that from sales of publications the Society can expect to receive a considerable portion of the difference. The cost of the portrait and of testimonials to Professor T. S. Fiske was met by special subscriptions, accounted for outside of the funds of the Society. Of the $1,500 set aside for other purposes of the Semicentennial, there remains a balance of $1,156.71.

The Organizing Committee of the Congress reported on a variety of matters such as arrangements with the United States Government to send out invitations to foreign governments, plans for inviting delegates, a list of twenty-five persons to be invited to give hour addresses at the Congress, arrangements for notifying universities about possible foreign lecturers for the summer of 1940, and the amount of $27,500 pledged for carrying on the work of the Congress.

A committee consisting of Professor C. R. Adams (chairman), Dean G. D. Birkhoff, Professor A. B. Coble, Dr. T. C. Fry, Professors Marston Morse and G. T. Whyburn was appointed to ascertain whether the time is favorable for starting an abstract journal in America under the auspices of the Society, whether international cooperation for such a venture can be obtained, and whether money is available from sources outside the Society for an initial period of five years. If studies show that conditions in these respects are favorable, the committee has power to proceed promptly to set up such a journal for an experimental five-year period.

A petition was received from a group of members of the Society, requesting changes in the by-law regarding election of members of the Council.

Professor Theodore von Kármán of the California Institute of Technology was invited to deliver the Josiah Willard Gibbs Lecture in December, 1939, in connection with the Annual Meeting.

The Cole Prize in Algebra is to be awarded at the Annual Meeting

of 1939 to the author of a paper in Algebra published during the years 1934–1938.

Titles and cross references to the abstracts of papers read at the regular sessions follow below. The papers were read as follows: papers numbered 1 to 7 in the section for Analysis on Wednesday afternoon; papers 8 to 12 in the section for Topology and Geometry on Wednesday afternoon; papers 13 to 20 in the section for Analysis on Thursday morning; papers 21 to 28 in the section for Calculus of Variations and Applied Mathematics on Thursday morning; papers 29 to 37 in the section for Algebra on Thursday morning; papers 38 to 46 in the section for Analysis on Friday morning; papers 47 to 54 in the section for Topology on Friday morning; and papers 55 to 95, whose abstract numbers are followed by the letter *t*, were read by title. Of those presenting papers, Mr. C. J. Blackall was introduced by Dr. D. C. Lewis, Dr. Albert Heins by Mr. H. F. S. Jonah, Mr. M. H. Heins by Professor J. L. Walsh, and Mr. Venable Martin by Professor J. H. Roberts.

1. C. J. Blackall: *Volume integral invariants of non-holonomic systems.* (Abstract 45-1-3.)

2. F. C. Smith: *Relations among the fundamental solutions of the generalized hypergeometric equation when* $p=q+1$. II. *Logarithmic cases.* (Abstract 45-1-62.)

3. Fritz John: *An identity for the integrals of a function over two concentric ellipsoids.* (Abstract 45-1-27.)

4. R. H. Cameron and Norbert Wiener: *Convergence properties of analytic functions of Fourier-Stieltjes transforms.* (Abstract 45-1-8.)

5. C. O. Oakley: *On the representation of line segments in the plane by equalities.* (Abstract 45-1-48.)

6. Israel Halperin: *The spectrum of a continuous flow.* (Abstract 45-1-19.)

7. Kurt Gödel: *The consistency of the generalized continuum-hypothesis.* (Abstract 45-1-108.)

8. G. E. Schweigert: *Concerning the hyperspace associated with a pointwise periodic transformation.* (Abstract 45-1-58.)

9. L. M. Blumenthal and G. R. Thurman: *Spheroidal and pseudo-spheroidal spaces.* (Abstract 45-1-4.)

10. L. A. Dye: *Involutorial transformations determined by two systems of lines.* (Abstract 45-1-12.)

11. G. B. Price: *Spaces whose elements are sets.* (Abstract 45-1-54.)

12. D. W. Hall (National Research Fellow): *On pointwise periodic homeomorphisms in the plane.* Preliminary report. (Abstract 45-1-87.)

13. O. E. Lancaster: *Orthogonal polynomials defined by difference equations.* (Abstract 45-1-32.)

14. John von Neumann and I. J. Schoenberg: *Fourier integrals and metric geometry.* II. (Abstract 45-1-47.)

15. J. A. Shohat and J. D. Tamarkin: *On mechanical quadrature formulas for infinite intervals.* (Abstract 45-1-61.)

16. L. R. Ford: *Normal families and differential equations.* (Abstract 45-1-14.)

17. H. M. MacNeille: *A constructive theory of integration.* (Abstract 45-1-41.)

18. W. T. Scott and H. S. Wall: *Properties of analytic functions derived from their corresponding continued fractions.* (Abstract 45-1-59.)

19. Otto Szász: *On the Cesàro and Riesz means of Fourier series.* (Abstract 45-1-66.)

20. L. B. Hedge: *Integration in a compact metric space.* (Abstract 45-1-88.)

21. Everett Pitcher: *The critical points of a map on a circle.* (Abstract 45-1-53.)

22. K. O. Friedrichs and J. J. Stoker: *The nonlinear boundary value problem of the buckled plate.* (Abstract 45-1-15.)

23. E. J. McShane: *The Jacobi condition and the index theorem in the calculus of variations.* (Abstract 45-1-38.)

24. M. H. Martin: *Restricted problems in three bodies.* Preliminary report. (Abstract 45-1-43.)

25. F. L. Lamoreau: *Temperature distribution in a long rectangular bar of two layers.* (Abstract 44-9-393.)

26. Ruth R. Struik and Miriam van Waters: *A mathematical method applied to criminology.* (Abstract 45-1-64.)

27. Alfred Korzybski: *Extensionalization in mathematics, mathematical physics, and general education. General semantics.* II. (Abstract 45-1-31.)

28. Harald Bohr and D. A. Flanders: *Algebraic functions of analytic almost periodic functions.* (Abstract 45-1-79.)

29. O. F. G. Schilling: *A remark on normal extensions.* (Abstract 44-11-473.)

30. Charles Hopkins: *Rings with minimal condition for left ideals.* (Abstract 45-1-25.)

31. B. W. Jones: *A note on a minimum for positive quadratic forms.* (Abstract 45-1-28.)

32. Saunders MacLane: *Modular fields.* I. *Separating transcendence bases.* (Abstract 45-1-39.)

33. W. A. Manning: *A note on transitive groups with regular subgroups of the same degree.* (Abstract 45-1-42.)

34. A. J. Kempner: *On the roots of algebraic equations with complex coefficients.* (Abstract 45-1-30.)

35. Alvin Sugar: *On a theorem in additive theory of numbers.* (Abstract 45-1-65.)

36. H. H. Campaigne: *Hypergroup structures.* (Abstract 45-1-9.)

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39. S. E. Warschawski: *On functions analytic in a half-plane.* (Abstract 45-1-73.)

40. W. J. Trjitzinsky: *Some general developments in the theory of functions of a complex variable.* (Abstract 44-11-475.)

41. Marston Morse and G. A. Hedlund: *Symbolic dynamics.* II. *Sturmian sequences.* (Abstract 45-1-46.)

42. C. N. Moore: *On a new definition for derivatives of non-integral order.* (Abstract 45-1-45.)

43. Dunham Jackson: *Linear dependence of polynomials on algebraic curves in space.* (Abstract 45-1-26.)

44. S. M. Ulam: *On the abstract theory of measure.* (Abstract 45-1-68.)

45. Nelson Dunford and B. J. Pettis: *On completely continuous operations in L.* (Abstract 45-1-85.)

46. Richard Courant: *Conformal mapping of multiply connected domains.* (Abstract 45-1-11.)

47. J. L. Kelley: *Fixed sets under homeomorphisms.* (Abstract 44-11-469.)

48. L. W. Cohen: *On imbedding a space in a complete space.* (Abstract 45-1-10.)

49. G. T. Whyburn: *Semi-locally-connected sets.* (Abstract 45-1-75.)

50. J. C. Oxtoby: *Note on metrically transitive transformations.* Preliminary report. (Abstract 45-1-51.)

51. C. C. Torrance: *Topologies in which the operation of closure is continuous.* (Abstract 45-1-67.)

52. R. G. Lubben: *Composition points in abstract spaces.* (Abstract 45-1-34.)

53. F. B. Jones: *Concerning the boundary of a complementary domain of a continuous curve.* (Abstract 44-11-468.)

54. A. D. Wallace: *Connected coverings and monotone transformations.* (Abstract 45-1-69.)

55. V. W. Adkisson: *Closed disconnected sets with unique maps in the plane.* (Abstract 44-11-463-*t*.)

56. A. A. Albert: *New proofs of the main theorems on algebras.* (Abstract 45-1-1-*t*.)

57. E. F. Beckenbach and Maxwell Reade: *A characterization of isothermic spherical maps.* (Abstract 45-1-76-*t*.)

58. R. P. Boas (National Research Fellow): *The closure of translations of almost periodic functions.* (Abstract 45-1-5-*t*.)

59. R. P. Boas (National Research Fellow) and Salomon Bochner: *On a theorem of M. Riesz for Fourier series.* (Abstract 45-1-6-*t*.)

60. J. W. Calkin: *An abstract self-adjoint boundary value problem.* (Abstract 45-1-7-*t*.)

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62. Jesse Douglas: *The analytic prolongation of a minimal surface over a rectilinear segment of its boundary.* (Abstract 44-11-465-*t*.)

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64. H. L. Garabedian: *A sufficient condition for Cesàro summability.* (Abstract 45-1-86-*t*.)

65. Saul Gorn: *The complete envelope of a non-normal partially ordered set.* (Abstract 45-1-17-*t*.)

66. Saul Gorn: *Some theorems on dimensions in partially ordered sets.* (Abstract 45-1-16-*t*.)

67. D. W. Hall (National Research Fellow): *Some applications of true secondary elements.* (Abstract 45-1-18-*t*.)

68. O. G. Harrold: *On the non-existence of certain continuous transformations defined on an arc.* (Abstract 45-1-20-*t*.)

69. Albert Heins: *Some remarks on the solution of partial difference equations.* (Abstract 45-1-21-*t*.)

70. M. H. Heins: *Interpolation to functions harmonic and positive or bounded in the unit circle.* (Abstract 44-11-466-*t*.)

71. M. R. Hestenes: *Generalized problem of Bolza in the calculus of variations.* (Abstract 45-1-22-*t*.)

72. M. R. Hestenes and W. T. Reid: *A note on the Weierstrass condition in the calculus of variations.* (Abstract 45-1-23-*t*.)

73. T. R. Hollcroft: *Cuspidal limits for plane curves.* (Abstract 45-1-24-*t*.)

74. F. B. Jones: *Concerning simple linear Moore spaces and simple continuous curves.* (Abstract 44-11-467-*t*.)

75. J. L. Kelley: *Concerning a decomposition of compact continua.* (Abstract 45-1-29-*t*.)

76. E. H. Larguier: *A matrix theory of n-dimensional measurement.* (Abstract 45-1-33-*t*.)

77. Walter Leighton: *New convergence theorems for continued fractions.* (Abstract 44-11-470-*t*.)

78. Norman Levinson: *General gap Tauberian theorems.* II. (Abstract 44-11-471-*t*.)

79. Norman Levinson: *On functions of minimal exponential type.* (Abstract 44-11-472-*t*.)

80. W. H. McEwen: *Note concerning a generalization of Bernstein's theorem.* (Abstract 45-1-35-*t*.)

81. E. J. McShane: *Curve-space topologies associated with variational problems.* (Abstract 45-1-36-*t*.)

82. E. J. McShane: *On the uniqueness of the solutions of differential equations.* (Abstract 45-1-37-*t*.)

83. Saunders MacLane and O. F. G. Schilling: *Infinite number fields with Noether ideal theories.* (Abstract 45-1-40-*t*.)

84. Venable Martin: *Monotone transformations of non-compact 2-dimensional manifolds.* Preliminary report. (Abstract 45-1-44-*t*.)

85. Rufus Oldenburger: *Minimal numbers.* (Abstract 45-1-49-*t*.)

86. Oystein Ore: *A remark on the normal decompositions of groups.* (Abstract 45-1-50-*t*.)

87. W. T. Puckett: *A fixed cyclic element theorem.* (Abstract 45-1-55-*t*.)

88. W. T. Reid: *Isoperimetric problems of Bolza in non-parametric form.* (Abstract 45-1-56-*t*.)

89. L. B. Robinson: *On an equation solved by a fonctionnelle.* (Abstract 45-1-94-*t*.)

90. A. R. Schweitzer: *On the construction of "configurational" sets of ordered dyads in the foundations of geometry.* (Abstract 44-11-474-*t*.)

91. J. A. Shohat: *On a differential equation for orthogonal polynomials.* (Abstract 45-1-60-*t*.)

92. A. D. Wallace: *On interior transformations.* (Abstract 45-1-70-*t*.)

93. J. L. Walsh: *Maximal convergence of sequences of rational functions.* (Abstract 45-1-71-*t*.)

94. J. L. Walsh: *Note on the location of zeros of the derivative of a rational function whose zeros and poles are symmetric in a circle.* (Abstract 45-1-72-*t*.)

95. G. T. Whyburn: *On irreducibility of transformations.* (Abstract 45-1-74-*t*.)

T. R. Hollcroft,
*Associate Secretary*

## DERRICK NORMAN LEHMER—IN MEMORIAM

After thirty-seven years of service in the department of mathematics at the University of California, Professor Derrick Norman Lehmer died on September 8, 1938. He came to the University of California in 1900, just after receiving the degree of Doctor of Philosophy at the University of Chicago. All of his academic career was therefore spent at California. He retired in 1937, apparently in robust health and with plans and ambitions for a life of continued activity with numerous interests, to all of which he had productively contributed.

Born in Indiana, July 27, 1867, he graduated in 1893 from the University of Nebraska, from which he also received the degree of Master of Arts in 1896, and the honorary degree of Doctor of Science in 1932. He married Clara Eunice Mitchell in 1900, and three daughters and two sons were born to them. One of the sons, D. H. Lehmer, followed in the footsteps of his father and is now an assistant professor of mathematics at Lehigh University.

Apart from mathematics he evidenced great interest and ability in both poetry and music. This was shown by his membership in numerous literary organizations accorded only to those who had indicated productive ability; by his editorship of the University of California Chronicle, devoted to the publication of articles of a literary character; and by numerous contributions of his own, mostly in the field of poetry.

His membership in honorary and learned societies included Phi Beta Kappa, Sigma Xi, the American Mathematical Society, the Mathematical Association of America, the Circolo Matematico di Palermo, and the American Association for the Advancement of Science, of which he was a fellow.

For a number of years he devoted much of his spare time to creative music, and was particularly interested in the music of western Indians, producing two operas based on Indian themes, "The Harvest" and "The Necklace of the Sun." In addition to these he wrote and produced a number of songs based on Indian legend and musical themes.

His earlier mathematical interests lay in the field of synthetic projective geometry and the theory of numbers; indeed, the latter subject continued to occupy Lehmer's attention throughout his entire career.

He found in rational numbers a fascination which lasted throughout his life. Primes, residues, quadratic forms, divisors of forms, continued fractions and their generalizations were seldom absent from his thoughts. Above all, the problem of factoring never failed to exert its appeal. The abstruse results of higher theory can be made to contribute to that tantalizing problem, though not yielding a complete solution. Such a situation was of the kind to especially attract him, and a large number was always a challenge. The possibility of experimental verification of a theorem or of a conjecture so often present in the theory of numbers led him to seek to increase the material available for such verification and so to extend and correct the factor table. This was the largest and most finished of his undertakings.

His table giving the least divisors of the first ten million numbers is a monument of accuracy and judgment. Others, notably Dase, Burckhardt, Glaisher, had labored successfully and had set a standard which he surpassed. This work occupied his available time for about nine years. Lehmer was not a computer, such as were Dase or Burckhardt, but like Glaisher, a mathematician, alive to the significance of theory and viewing the table as a contribution to it.

If an irrational number is developed in a continued fraction, the convergents have well known properties regardless of which irrational number is developed. But if the irrational is some distinguished number, the convergents may be expected to have special properties. To find these is a difficult problem toward the solution of which little has been done. An example may serve. Cotes found the continued fraction for $e$ to be $2+1/1+1/2+1/1+1/1+1/4+1/1+\cdots$ where there are triads 1, $2k$, 1 in the quotients. Lehmer's result is this: the denominators of the convergents of orders $3n$, $3n-2$, $3n-6$ and the numerator of order $3n-3$ are divisible by $n$. Continued fractions whose quotients present a regularity of the type exemplified by $e$ had been studied by Hurwitz, and Lehmer was able to extend his results to the point where particular theorems such as the one just given may be obtained.

Continued fractions may be generalized in different directions. Of these generalizations one due to Jacobi is interesting. Triads of numbers are determined by relations of recurrence just as pairs are determined in the case of the simple continued fraction. If there is periodicity in the recurrence relations, the triads may furnish convergents to the roots of a certain cubic equation, but they do not always do so. The distinction required elucidation which was given in several papers that cleared up the matter completely.

To analytical number theory an important contribution was made by Lehmer. A special result must suffice, but one typical of a wide class. Let $\theta(x)=1$ if all the prime divisors of $x$ are of the form $4n+1$, but otherwise let $\theta(x)=0$. Let $V(x)$ denote the number of distinct prime divisors of $x$, then the quotient $(1/N)\sum_1^N 2^{V(x)}\theta(x)$ tends to a limit when $N$ increases. The limit is $1/\pi$. Such determinations must be made if answers to many simple questions are required. Quotients similar to the above do not always have limits. Lehmer was able to obtain definitive results in important cases. His conjectures in the cases he was not able to treat fully were examined by Landau.

In the contrast to number theory which geometry offers, Lehmer often found relaxation. He taught projective geometry with enthusiasm and wrote a text introductory to the subject. His own contributions were directed to the application of synthetic methods to less usual topics such as the unicursal cubic, the covariant conic of a pencil, quadratic transformations. His insistence was on a solution in the fewest possible steps so that constructions should not only be theoretically possible but might really be effected. Variety of interest was added by studies on the tetrahedroid, planimetric devices, the quadratic complex, the spiral of Cornu.

He was a distinguished teacher with power to kindle enthusiasm not only in the classroom, but in his occasional lectures, when he presented the wonders of number theory, or exhibited curves in space traced by lights in a dark room by means of a mechanism devised by one of his students, Dr. Saul Pollock. He supervised personally, spending freely of time and thought, the production of more than twenty theses for the doctorate.

The variety of his mathematical interests is well shown in the appended bibliography which is arranged chronologically.

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T. M. PUTNAM

## JEFFREYS ON SCIENTIFIC INFERENCE

*Scientific Inference.* By Harold Jeffreys. Cambridge, University Press, 1937. 6+272 pp.

This book was first published in 1931, and is now reissued, with addenda. Its beginnings were in a series of papers published by the author and Dorothy Wrinch in the "Philosophical Magazine" and in "Nature" during the years 1919 to 1923. Professor Jeffreys, of St. John's College, Cambridge, ventures here into the nature of scientific inquiry, and especially into what he considers the chief guiding principle of both scientific and everyday knowledge: that it is possible to learn from experience and to make inferences from it beyond the data directly known by sensation. These inferences themselves, however, are subjected to change. Fundamental scientific laws have been found inaccurate. For twenty years physical science has been modifying and reconstructing its laws as a result of new knowledge. The reconstruction has followed the old method, but will this always be possible? Have recent developments shown that scientific method itself is open to suspicion?

Jeffreys attempts in his book to place scientific method above suspicion. "There is no more ground now than thirty years ago for doubting the general validity of scientific method, and there is no adequate substitute for it. When we make a scientific generalization we do not assert the generalization or its consequences with certainty; we assert that they have a high degree of probability on the knowledge available to us at the time, but that this probability may be modified by additional knowledge. The more facts are shown to be coordinated by the law, the higher the probability of that law and of further inferences from it. But we can never be entirely sure that additional knowledge will not some day show that the law is in need of modification. The law is provisional, not final; but scientific method provides its own means of assimilating new knowledge and improving its results" (pp. 6–7).

This reasoning therefore depends on the introduction of probability considerations. "The notion of probability, which plays no part in logic, is fundamental in scientific inference. But the mere notion does not take us far. We must consider what general rules it satisfies, what probabilities are attached to propositions in particular cases, and how the theory of probability can be developed so as to derive estimates of the probabilities of propositions inferred from others and not directly known by experience" (p. 7).

The notion of probability is established as a relation between a proposition and a set of data, intelligible to everyone who ever says, "I am nearly sure of that," or similar expressions. Every probability can be associated with a real number from 0 to 1. A symbolism can be developed starting with $P(p|q)$ for the probability of the proposition $p$ on the data $q$. If $p_1, p_2, \cdots, p_n$ are a number of mutually exclusive hypotheses such that one of them must be true, and if $P(p|q \cdot h)$ is the probability that $p$ is true given $h$ and $q$, then Jeffreys' central theorem is Bayes' law:

$$P(p_r|q \cdot h) = \frac{P(q|p_r \cdot h)P(p_r|h)}{\sum_{r=1}^{n} P(q|p_r \cdot h)P(p_r|h)}.$$

"This law," writes the author, "is to the theory of probability what Pythagoras' theorem is to geometry" (p. 19).

Bayes' law enables us to pass from a priori probabilities, through superior knowledge, to new, a posteriori probabilities. We must start somewhere, with a prior probability. Scientific work tends to make the posterior probability so near zero or unity as to amount to practical certainty. This contention is exemplified by an exposition

of the theory of sampling. In the process of discovery of quantitative laws, as in physics, we cannot apply the general principle without some further specification. This is found in the so-called simplicity postulate. Jeffreys establishes it in the form: "Every quantitative law can be expressed as a differential equation of finite order and degree, in which the numerical coefficients are integers" (p. 45). Such a law restricts the number of admissible laws to $\aleph_0$ and the a priori probabilities must decrease with decreasing simplicity.

Now follows an investigation of the theory of errors. There is a belief that there are true values of the quantities to be measured, and that the observed values have certain errors. The existence of true values is a postulate, and its validity has to be determined. This leads to a derivation and discussion of the normal law of errors, and of different types of errors.

The remaining part of the book is mainly an application to various fields of theoretical physics of the principles enunciated in the first chapters. They deal with the measurement of physical magnitude, mensuration (distance and angle), Newtonian dynamics, light and relativity, and contain careful studies of the different steps necessary in building the concepts and the equations, and their justification by the available evidence. In regard to the theory of relativity we meet the conclusion that there is no antagonism between the principle of relativity and the simplicity postulate. The general theory of relativity is justified as a general law up to a certain point, and the simplicity postulate entitles us to extend it further, if possible.

In the last chapter we find a discussion of miscellaneous questions. First comes the topic: "Is there a non-quantitative simplicity postulate?" The question is answered in the affirmative. The principle seems to be that if an object of a given class has $r$ properties $a, b, c, \cdots, k$, then there is a finite a priori probability that all future members of the class with any $r-1$ of these properties will also have the remaining one. This principle guides the botanist in determining a new species. Under "Ultimate concepts" the question is discussed whether the process of constructing concepts of increasing generality will ever stop. This is denied in the case of the theory of quanta, and that of life. There are also observations on the subconscious, on determinism and causality. Another question is whether "neglecting small quantities" and arguing by "orders of magnitude" can be taken as a departure from popular standards of accuracy.

In Chapter 11 the author compares his theory of scientific knowledge to other theories, notably the statistical theory of probability (Venn), Keynes' theory, Russell's theory of sense data, Whitehead's theory of events. In an appendix we find some data on probability in logic and pure mathematics, on transfinite numbers, and the analytic treatment of sine and cosine. In an "Addenda" Jeffreys gives us a summary of new results, in which his theory now can decide whether a set of observed data support or do not support the introduction of a new parameter to express them.

The book of Jeffreys is an interesting and valuable contribution to the whole problem of induction. The assertions of philosophers on this subject are seldom substantiated to such an extent by a large number of examples from many branches of natural science. The systematic way in which the author, step after step, analyzes the different experiences which justify the theoretical foundations of the sciences is quite valuable for a fundamental understanding. The student of the theory of samples and errors, of dynamics, of light, and of relativity will find much useful and stimulating material in this book, even if he has his doubts about Jeffreys' specific theory of scientific inference.

To the reviewer, at any rate, such doubts exist. He thoroughly agrees with the

author's materialistic point of view, which rejects the doctrine that a physical object is nothing but a class of sensibilia. "The physical object," he argues against Russell, "and the laws of physics are anterior in knowledge to the sensibilia" (p. 226). In other words, science deals with an objective world. Jeffreys' theory now seems to state that all our scientific laws are more or less probable expressions of objective laws. It seems doubtful, however, whether it is correct to introduce the concept of probability for expressing the fact that our laws are approximations of laws in nature. Probability is a concept of natural philosophy too, expressing an objective phenomenon. It is mathematically expressed by the theory of measure, and physically realized in statistical behavior. It is a notion concerning the behavior of groups, and as such expresses facts which are not necessarily enforced for the individual. It is not clear what the meaning of probability, thus conceived, can be for the validity of, say, Newton's law. For Jeffreys, however, probability is another theory. It expresses a relation between a proposition and a set of data. Probability expresses a judgment.

This theory of scientific inference is therefore a theory of judgment, and introduces a subjective element into the expression of the laws of nature. It seems to the reviewer that the certainty of obtaining objective knowledge by induction can only be obtained by eliminating the subjective element from our observations, and not by applying a method which itself brings in a subjective element. Somehow the question: How probable are Maxwell's equations? seems to miss the point. But even if we may have our doubts concerning the success of Jeffreys' efforts in the theory of knowledge, we must agree at the same time that the careful way in which he has built up his case allows us to study in detail its different implications.

D. J. Struik

## SHORTER NOTICES

*Über einige neuere Fortschritte der additiven Zahlentheorie.* By Edmund Landau. (Cambridge Tracts in Mathematics and Mathematical Physics, no. 35.) London, Cambridge University Press; New York, Macmillan, 1937. 94 pp.

This tract presents the last published book of the great mathematician, and reflects in crystal clear fashion many characteristic traits of his mathematical work: an exceptional ability to refine the arguments of workers in the same field, anatomical exposure of the pattern of proof, elimination of detours, a tendency toward elementary methods (not at all synonymous with simple), the reduction of the use of words to a very minimum. The result is a completely self-contained booklet (with the exception of the *Anhang* on Siegel's theorem on quadratic forms of negative discriminant) which could (theoretically) be read by any graduate student after a course in elementary number theory and advanced calculus. In reality, the tract represents very difficult reading, on account of its extreme conciseness.

It is interesting that Landau has in this work completely eliminated the use of his $O(\ )$, $o(\ )$, using instead a set of a hundred thirty-five $c_\gamma$'s.

After a very condensed introduction of nine pages, in which the aims of the various chapters are outlined, follow four chapters and an appendix: 1. Der Winogradoffsche Satz, 30 pp. 2. Der Schnirelmannsche Satz, 20 pp. 3. Die Sätze von Erdös und Romanoff, 11 pp. 4. Sätze von Khintchine, Besicovitch und I. Chowla, 11 pp. Anhang. Der Siegelsche Satz, 13 pp.

The object of Chapter 1 is to prove Winogradoff's theorem: $\lim_{k=\infty} G(k)/k \log k \leqq 6$, where $G(k)$ is the smallest number sufficient to represent all sufficiently large positive integers as the sum of $G(k)$ or fewer $k$th powers of positive integers.

This theorem of 1934 is a landmark in the history of Waring's problem, which was first proposed about a century and a half ago, and to which elementary methods (in the ordinary sense of the word) were applied until Hilbert, in 1909, first proved the finiteness of $G(k)$. However, his analytic method, which involved the use of 5-fold Dirichlet integrals (first, 25-fold integrals), yielded a pure existence proof, and his methods, much as they excited the admiration of mathematicians, have not proved capable of such extension as was at the time hoped. Then followed the incredible applications of the theory of analytic functions by Hardy and Littlewood which first led to a definite large bound for $\lim G(k)$. From Winogradoff's theorem it is known that every integer $x$, sufficiently large, can be represented by $G(k)$ $k$th powers, where $k < G(k) < 6k \log k$.

The contributions of American mathematicians to Waring's problem, particularly those of Dickson and his students, are well known.

The proof of Winogradoff's theorem is broken down, in a manner familiar to readers of Landau's former books, into a sequence of sixty-eight theorems, of which the last is the theorem itself. The theorem is quoted as (1) in the Introduction, and it is a typical Landau touch to end this chapter after thirty pages with the two lines:

Satz 68. (1).

Beweis. Klar.

The two theorems of Schnirelmann which comprise the second chapter are as follows:

THEOREM 91. *Let $M(x)$ be the number of positive integers $y \leqq x$ for which $y = p + p'$ ($p$, $p'$ primes) has solutions. Then, for $x \geqq 4$, $M(x) \geqq x/c_8$, ($c_8$ a constant).*

THEOREM 95. *Every positive integer $x > 1$ can be represented as a sum of at most $c_6$ primes, ($c_6$ a constant).*

Hardy and Littlewood had earlier derived the result that every sufficiently large odd number is the sum of three odd primes, but under the assumption of an unproved theorem concerning the distribution of zeros of a certain transcendental function. Since Schnirelmann, Winogradoff has proved the Hardy-Littlewood theorem without any unproved assumption, thus including both the Hardy-Littlewood theorem and Schnirelmann's result. This is the present status of Goldbach's conjecture that every even number greater than 4 is the sum of two odd primes.

The theorems of Chapters 3 and 4 are of a newer and less well known type than those of the first two chapters. An illustration is the following theorem:

THEOREM 96 (Khintchine-Erdös). *Let $\mathfrak{A}$ be a set of positive integers $a$, and let $\mathfrak{B}$ be a set of positive integers $b$ enjoying the property that a certain positive integer $l$ exists such that every positive integer is a sum of $l$ or fewer numbers $b$. Let $\mathfrak{S}$ be the set of all numbers of the form $a$ or $a+b$, $A(x)$ the number of $a$'s $\leqq x$, $S(x)$ the number of $s$'s $\leqq x$, and, for some $\alpha$, $(0<\alpha<1)$, $A(x)\geqq\alpha$ for all $x$ greater than zero. Then, for all $x>0$, $S(x)\geqq\alpha(1+(1-\alpha)/2l)x$.*

The proof of this theorem is extremely short and involves a minimum of mathematical prerequisites. It gives almost the impression of an exercise in formal logic.

By specializing the set $\mathfrak{B}$, many important results are obtained, such as:

*For a given positive integer $a>1$ and for a positive integer $x>2$, the number of solutions of $p+i^a\leqq x$, $p$ a prime, for any integer $i>0$, is greater than $x/c(a)$, where $c(a)$ is a constant dependent only on $a$; and a similar theorem for $p+a^i\leqq x$.*

To another theorem by Khintchine, Theorem 110, which belongs to the same general range as Theorem 96, Landau pays the compliment: "Der Kintchinesche Beweis ist elementar und doch ein sehr kompliziertes grosses Kunstwerk."

This book is a fitting memorial to a mathematician who was at the same time a leader in research and unsurpassed in his ability and eagerness to make the most advanced work done by others available to a larger group.

A. J. KEMPNER

*Zur Differentialgeometrie der Kurven und Flächen fester Breite.* By Hans Bückner. (Schriften der Königsberger gelehrten Gesellschaft, vol. 14, no. 1.) Halle, Niemeyer, 1937. 22 pp.

This pamphlet gives a readable and very brief account of this subject, based on the support function of the curve or surface. New results include lower bounds for the area of a surface of constant breadth and for length of a space curve of constant breadth; characterization of a curve of constant breadth as an oval for which any two of the support normals intersect within the closed region bounded by the curve; an analogous result for surfaces; and an equation connecting lengths and areas of the two parts into which a support normal divides a curve of constant breadth and the included area.

E. H. CUTLER

*The Theory and Construction of Nondifferentiable Functions.* By A. N. Singh. (Lucknow University Studies, no. 1, Faculty of Science, Session 1934–1935.) Lucknow, Newul Kishore Press, 1935. 7+110 pp.

The author of this tract has published numerous papers on nondifferentiability and curves without tangents since 1924, mostly in Indian periodicals, though a couple of papers have found their way to this country. In the present volume, which is based

upon four lectures delivered at Lucknow University, he gives a synopsis of the work done in this field. The first lecture contains a brief history of the earlier attempts to construct nondifferentiable functions leading up to the example of Weierstrass. He discusses the method of Dini and the various series definitions, including the recent one due to van der Waerden. In the second lecture he treats functions defined geometrically, in particular, the curves of Bolzano, von Koch, Peano, Hilbert, Kaufmann, and Besicovitch, whereas the general method of Knopp is only mentioned. The third lecture contains his own arithmetical definitions which attach in part to earlier work of Peano and E. H. Moore. The last lecture deals with various properties of the derived numbers of nondifferentiable functions. In closing, let me add that anybody who is giving a course in real variables will find this little tract useful.

EINAR HILLE

*Moderna Teoria delle Funzioni di Variabile Reale.* Part 2. *Sviluppi in Serie di Funzioni Ortogonali.* By G. Vitali and G. Sansone. (Monografie di Matematica Applicata per cura del Consiglio Nazionale delle Ricerche.) Bologna, Zanichelli, 1935. 6+310 pp.

This second volume is a very valuable continuation of the first part which was reviewed in vol. 43 (1936), p. 15, of this Bulletin. It is devoted to expansions in orthogonal series and takes up quadratically integrable functions, Fourier series, Legendre series, Laguerre and Hermite series, and the Stieltjes integral. The first chapter gives the primary notions on Hilbert space, orthogonality, linear independence, approximation, convergence in the mean, and expansions in orthogonal series including the closure theorem of Vitali which serves as the basis for the discussion in the special cases. This is followed by a discussion of Fourier series, including convergence in the mean, local convergence criteria, Fejér and Poisson summability, and the Fourier integral (Fourier transforms are just mentioned). The treatment of Legendre series starts with an adequate discussion of the basic properties of the Legendre polynomials through the asymptotic formulas and leads up to Hobson's convergence theorem. The fourth chapter, dealing with Laguerre and Hermite series, is perhaps the most valuable in the whole book since these series are normally not discussed in standard texts on analysis. It presents the fundamental properties of the polynomials, including their asymptotic behavior, and ends with the convergence theorems of Stone and of Uspensky for Hermite series. The last chapter, which has very little contact with the rest of the book, gives a discussion of the Stieltjes integral with applications to the theory of distribution functions and their characteristic functions. There is a large bibliography. The treatment is up-to-date, rigorous without being heavy, and the book can be strongly recommended to those who have to give courses in real variables or classical mathematical physics.

EINAR HILLE

*Summable Series and Convergence Factors.* By C. N. Moore. (American Mathematical Society Colloquium Publications, vol. 22.) New York, American Mathematical Society, 1938. 6+105 pp.

Fairly early in the development of the theory of summability of divergent series, the concept of convergence factors was recognized as of fundamental importance in the subject. One of the pioneers in this field was C. N. Moore, the author of the book under review. He first introduced the name "convergence factors" in this connection, published some of the first convergence factor theorems, and has been one of the chief investigators in the subject since then. It is therefore appropriate that the first

systematic treatment of convergence factor theory in book form should have been written by him.

The author points out that all methods for the summation of a divergent series which have come into general use may be classified as mean-value methods or convergence-factor methods. In either case the object of the summability method is to determine a value or "generalized sum" for the series. Although convergence factors were first applied to summable series, it was soon seen that they could yield valuable information when applied to convergent series.

Moore classifies convergence factors into two types. In type I he places the factors which have only the property that they preserve convergence for a convergent series or produce convergence for a summable series. In type II he places the factors which not only maintain or produce convergence but have the additional property that they may be used to obtain the sum or generalized sum of the series. This book gives a generalized systematic treatment of the theory of convergence factors of both types, for simply infinite series and for multiple series, convergent and summable. For summability the author uses the method of Nörlund means instead of the Cesàro method, giving more general results. Many of the theorems and methods given in this work are original and the proofs have not been published elsewhere.

The book opens with an Introduction in which the early history of the idea of summability of series is discussed, and the emergence of the concept of convergence factors is traced historically. Chapter 1 deals with convergence factors in convergent series. For convergence factors of type I, necessary and sufficient conditions are derived in order that a convergent series may still converge after convergence factors are introduced into its terms. For factors of type II, additional necessary and sufficient conditions are obtained in order that the series with convergence factors may converge to the sum of the original series. Corresponding results are derived for multiple series whose partial sums are bounded. These convergence factor theorems serve to furnish criteria for the regularity of convergence-factor definitions and mean-value definitions of summability; they may also be used to obtain results concerning relations of equivalence and inclusion of summability methods.

In Chapter 2, a brief discussion is given of Nörlund summability of series, simple and multiple. Chapter 3 then takes up, for summable simple series, the corresponding questions to those of Chapter 1, for convergence factors of both types; Chapter 4 handles the case of summable double series, and Chapter 5 that of summable multiple series of higher order. Results for the Cesàro method of summability are obtained as special cases. The final chapter is concerned with convergence factors in restrictedly convergent multiple series. The book closes with a rather full bibliography of the subject.

Many of the convergence factor theorems hitherto published in the literature appear as special cases of the general theorems here derived. This book makes considerable progress in the attempt to present a definitive form to the subject of convergence factors.

L. L. SMAIL

*Analyse Mathématique.* Vol. 1. *Analyse des Courbes, Surfaces et Fonctions usuelles. Intégrales simples.* By Paul Appell. 5th edition, completely revised by Georges Valiron. Paris, Gauthier-Villars, 1937. 8+395 pp.

The first edition of this work appeared in 1898 and, according to the preface, it reproduced the course which for three years Paul Appell had been offering at l'École Central des Arts et Manufactures. It covered the essential elements of mathematical

analysis from the standpoint of the applications to geometry, mechanics, and physics. In the early revisions of the text Appell made notable changes with a view to strengthening the preparation for the study of mechanics and physics. The fourth edition appeared in 1921 and, in addition to representing the course given at l'École Central, it included certain material which figured in the program of admission; the reader, however, was assumed to have a knowledge of the exponential and logarithmic functions, elementary differential calculus, complex numbers, and the elements of analytic geometry.

In presenting the first volume of the new edition, Georges Valiron has made numerous changes not only in the choice and order of the subject matter but also in the method of treatment. He reminds us that the book is not intended to be a mathematical encyclopedia, but rather a course of instruction which is progressively more difficult and introduces new notions only as they are needed. For example, complex numbers are not presented until very late in the course when the advantage of their use in certain problems is readily appreciated.

The first three chapters, which have been added by Valiron, serve as a geometric introduction and are based on lectures commonly given at the beginning of the course in mechanics at the Sorbonne. In these eighty pages the author introduces the vector notation and then develops the elements of plane and solid analytic geometry. Although Valiron's treatment is more extensive, it reminds one somewhat of Appell's *Éléments de la Théorie des Vecteurs et de la Géométrie Analytique.*

Beginning with the fourth chapter the new edition develops the present program of analysis and geometry for the certificate in general mathematics of the Faculty of Sciences of Paris. This program has been in use since 1931 and it preserves the spirit of Appell's work even though the general plan has been somewhat altered. The scope of this portion of the book is indicated by the chapter headings: 4. Fonctions d'une variable. Limites. Continuité. 5. Fonctions dérivables. 6. Fonctions primitives. Intégrales. Différentielles. 7. Fonctions exponentielle et logarithmique. 8. Méthodes d'intégration. 9. Intégrales définies dont une limite est infinie où portant sur une fonction non bornée. 10. Fonctions de plusieurs variables. 11. Courbes planes ou gauches. Courbure. Enveloppes. 12. Étude des courbes en coordonnées polaires. 13. Intégrales curvilignes. Calcul des aires et des volumes.

A comparison with the fourth edition shows that Valiron has rearranged the subject matter considerably. There are frequent additions; in particular, the chapter on exponential and logarithmic functions is new. Vector methods have simplified many of the discussions; this is especially noticeable in the study of plane and space curves. In this first volume very little use is made of infinite series. Numerous well-chosen examples from the fields of geometry and physics are solved in the text and much emphasis is placed on graphic representation and methods of approximation. The figures are numerous and very well drawn. The text is attractively printed and the number of typographical errors noted was small.

C. H. YEATON

*The Theory of Linear Operators from the Standpoint of Differential Equations of Infinite Order.* By H. T. Davis. Bloomington, Indiana, Principia 1936. 14+628 pp.

List of contents: 1. Linear operators. 2. Particular operators. 3. The theory of linear systems of equations. 4. Operational multiplication and inversion. 5. Grades defined by special operators. 6. Differential equations of infinite order with constant coefficients. 7. Linear systems of differential equations of infinite order with constant

coefficients—the Heaviside calculus. 8. The Laplace differential equation of infinite order. 9. The generalized Euler differential equation of infinite order. 10. Differential operators of infinite order of Fuchsian type—infinite systems. 11. Integral equations of infinite order. 12. The theory of spectra. Bibliography.

Professor Davis has written a number of papers on differential equations of infinite order. These papers he has now expanded into a voluminous treatise on linear operators. His own investigations form the core of Chapters 2 and 4 to 11 of the present book though he has incorporated much material from the work of other scientists. The reader will find much of stimulating interest in this treatise. The references are unusually complete and the reader who wants to learn something about the history of operational calculus, especially on the formal side, will find this monograph indispensable as a source book. As a matter of fact, one can get some information about almost any mathematical question by referring to the subject index of this book. Sometimes the information is excellent, a case in point being the treatment of Hadamard's determinant theorem on pp. 116–123 with its rich bibliography, but far too often the wording is ambiguous and sometimes misleading. It is a decided weakness of the book that the treatment is spread too thin over too vast an area. The author is inclined to include in the text excerpts from the literature, long quotations, theorems frequently stated without proper background, etc. The book would have gained had this material been severely pruned.

These are criticisms of detail. There are larger issues at stake. As is apparent from the title of the book, the author treats linear operators from the standpoint of differential equations of infinite order. It is quite natural that he should wish the power of this method to appear as strong as possible, but the claims that he makes for it strike the reviewer as insufficiently supported by the evidence presented in the book. His basic operation is differentiation, $z=d/dx$, and practically all his operators are power series in $z$ with coefficients which are analytic in $x$. The natural domain for such an operator is the field of analytic functions, and, frequently, the set must be narrowed down to entire functions of exponential type. Now it is well known that some differential operators of infinite order coincide in their proper domain of definition with operators having a much wider range of existence. In other words, the operator admits of an extension. The simplest case in point is the operator $\exp(hz)$ which, whenever it has a sense, transforms $f(x)$ into $f(x+h)$.

As far as I can see the author supports his claims of having a broad and powerful approach to the operational calculus upon three theses, all highly interesting. The first is a principle of extension: every (respectable) differential operator admits of an extension outside the domain of analytic functions. The second is a special case of a principle of the permanence of functional equations which could be formulated as follows. If the linear equation $y=Ux$ possesses an inverse $x=U^{-1}y$ when $y$ is suitably restricted and if this inverse transformation possesses an extension outside of its original domain of definition, then the extension is still an inverse of the original equation. The third is a related principle of formal invariance. If an inversion formula, derived under restrictive hypotheses on the operator, remains meaningful when these assumptions are removed, then the formula also remains an inversion formula for the general operator.

I do not think that the author has clearly formulated these principles anywhere in his book, but they underlie most of his work. I have no objections against these principles as working hypotheses, the results so obtained being verified directly, or better still, as strictly formulated and rigorously proved theorems. When in Chapter 11, the author sets out to derive the theories of the equations of Volterra and of Fredholm from differential operator theory, he is really invoking the second and third

of these principles, but in the absence of proofs of these principles, the value of this approach must be held in abeyance. That the extension principle is valid in special instances is documented by numerous important examples, but the justification in general cases is missing. It is impossible to overlook these defects in a book which otherwise has many good features.

EINAR HILLE

*An Introduction to Riemannian Geometry and the Tensor Calculus.* By C. E. Weatherburn. Cambridge, University Press; New York, Macmillan, 1938. 11+191 pp.

In the author's words this is "a book which will bridge the gap between the differential geometry of euclidean space and the more advanced work on the differential geometry of generalized space." It is dedicated to Dean Eisenhart and Professor Veblen. Indeed it follows very closely the content, notation and arrangement of the former's *Riemannian Geometry*. But it is purposely more elementary, less rigorous and less complete. Beside the basic essentials there are chapters on congruences and orthogonal ennuples, on hypersurfaces in euclidean and Riemannian space and on general subspaces. At no point does the book venture outside the domain of Riemannian geometry proper.

The following features are noteworthy:

(1) An introductory chapter giving a résumé of theorems of algebra and analysis used freely in the later chapters.
(2) A pleasantly restrained use of the notation of classical vector analysis as in the author's earlier *Differential Geometry of Three Dimensions.*
(3) Sections based on some original work by the author on quadric hypersurfaces.
(4) The very desirable use of R. Lagrange's generalized covariant differentiation in the treatment of subspaces.
(5) 125 exercises distributed among the chapter endings, some of which include important subject matter not in the text proper.
(6) An appended historical note condensed from an address (1932) by the author in which reference is made to the various generalizations of Riemannian geometry.
(7) A bibliography extending from the year 1827 with 132 entries and references to bibliographies by other authors.
(8) The excellent typography and general physical characteristics.

It will be a disappointment to some that the book was not constructed along more original and more stimulating lines although this is much to demand of an introductory text. Greater effort might have been made to lessen the emphasis on formalism which is so difficult to avoid in this field. Basic concepts could have been presented more carefully and given richer meaning. For example, increments and differentials are confused occasionally in accordance with a time-honored but deplorable custom; and the tensor concept as presented here is little more than a law of transformation. The notion of tangent space of differentials would be of assistance on both counts. Important existence theorems and elementary topological considerations would have been welcome. For example, the generalized Gauss-Codazzi equations are written down but their fundamental importance as existence conditions left unmentioned, and throughout the book the word "space" is never precisely defined. An obvious omission on the formal side is that of the alternating $\epsilon$-tensors.

The book is neither for dilettantes nor advanced students of the subject. But to those who are seeking an introductory treatment of textbook character to serve perhaps as a sequel to the author's *Differential Geometry* it is to be recommended.

J. L. VANDERSLICE

*Mathematical Biophysics. Physicomathematical Foundations of Biology.* By Nicolas Rashevsky. University of Chicago Press, 1938. 18+340 pp.

It is likely that biology will eventually be as full-panoplied with mathematically expressed theory as physics now is. The process is already started, and the history of the older natural sciences like astronomy, physics, and chemistry admits of no doubt as to the final outcome. There is no substitute for mathematics to state in rational shorthand the relations between natural phenomena or generalizations about them. Progress towards a sound and useful mathematical biology will be slower, however, than has been the case with either physics or chemistry. The phenomena are inherently more complex. What happens when pure chemical substances react together can be completely described, and predicted, in terms of the stable constitution of the initial substances plus the environing circumstances within the frame of which the reaction takes place. What living organisms do involves a third and extremely troublesome element. An organism is not a stable thing of fixed and permanent intrinsic constitution, so long as it is alive. On the contrary it is constantly changing internally, and these internal changes play an important part in determining what it will do in a given set of circumstances. Too little is known in detail about the precise pattern of these changes to make it possible now to reduce them to simple and rigorous mathematical expression, save in a comparatively few and relatively simple cases.

So then the approach to the laying of physicomathematical foundations of biology, towards which Rashevsky is pioneering in this book, has to be at present by the postulational route. In essence the procedure is to take some biological phenomenon, such as cell division or the conduction of a nervous impulse, and lay down a set of initial postulates about it, which shall be at once both simple enough to be manageable mathematically and complicated enough hopefully to have at least some relation to biological actuality, and then work out mathematically the logical consequences that will flow from the postulates and their interplay. This is the plan followed in the present book relative to such biological matters as cell morphology, cell division, diffusion of substances in and through cells, and various phenomena involving the nervous system, central and peripheral. It would be too much to expect at this early stage of development of the subject that there should be any coherent or unified "red thread" of important theory running through the book. There is none. Rather Rashevsky's present objective appears to be exploratory—trying out the general postulational technique on a variety of biological problems with the hope of uncovering some promising leads that will repay further and more penetrating treatment.

Somewhat unfortunately the pioneers in this field of endeavor have a hard and discouraging row to hoe. The reaction of the biologists—including both those who are able to understand the mathematical procedures and the much larger number who are not—is apt to be that the initial postulates are always too much simplified to have any *significant* relation to biological reality as they know it (and the mathematicians do not). On the other hand the mathematicians find little nourishment in the comparatively simple and elementary (to them) mathematical procedures employed. In the present book there are few mathematical ideas beyond those comprised in the general frame of theoretical mechanics—statics and dynamics. To be sure a few of the developments and results do suggest purely mathematical problems worth following further from that viewpoint. But in the main the manipulations along well-worn paths of algebra, calculus, and differential equations will not greatly excite competent mathematicians.

But in spite of this more or less inevitable falling between two stools that is the pioneer's sad lot, the endeavor in itself is altogether worthy of encouragement. Mathematical biology will never develop unless somebody starts the process. For-

tunately it is now well started with the work of Lotka, Volterra, and Rashevsky, not to mention various lesser but significant and useful workers. Mathematicians should read this book, if for no other reason than to get a first-hand acquaintance with the early steps of the Queen of the Sciences to bring under her dominion another great field of natural knowledge. Biologists should read it for the same reason, and also because a number of the theoretical developments suggest new lines of experimental approach to old and important unsolved problems.

RAYMOND PEARL

*The General Field Theory of Schouten and Van Dantzig.* By N. G. Shabde. (Lucknow University Studies, no. 10.) 1938. 4+58 pp.

During the 25 years that have elapsed since the appearance of the general relativity theory the subject has attracted the attention of many mathematicians and has been developed in several different directions. The pamphlet under review belongs to the series of rather formal developments which were prompted by the idea, as the author puts it, that "in Riemannian geometry of ordinary general relativity a unification of electromagnetic and gravitational phenomena is impossible"; whether the statement is true or not, this conviction has been the source of a series of brilliant and elegant generalizations connected among others with the names of Weyl, Cartan, Kaluza, Veblen. In particular, Shabde's contribution is more closely related to the work of Schouten and Van Dantzig. It consists of an exposition in somewhat generalized form of projective relativity of Veblen followed by reports on the author's two papers published in periodicals. The pamphlet hardly can be recommended for one who has not already acquired familiarity with the theories presented; for a specialist it may be convenient to have in this form the contents of the author's articles, one of which has appeared in a not easily accessible publication. Misprints are abundant and some may be disturbing for the uninitiated.

G. Y. RAINICH

*Topologie der Polyeder und kombinatorische Topologie der Komplexe.* By K. Reidemeister. (Mathematik und ihre Anwendungen, vol. 17.) Leipzig, Akademische Verlagsgesellschaft, 1938. 9+196 pp.

This book contains a rigorous and clear treatment of the combinatorial theory of polyhedra. The point of view is strictly combinatorial, there being no attempt to introduce notions involving continuity. A polyhedron is a subset of a linear $n$-space which is a sum of convex rectilinear pieces, the latter being intersections of a finite number of linear half-spaces and hyperplanes. By definition, two polyhedra are topologically equivalent if they have isomorphic subdivisions. A discussion of the simple transformations of Alexander (cf. Annals of Mathematics, (2), vol. 31 (1930), pp. 292–320) is given as is the theorem of Newman (cf. Proceedings of the Academy of Sciences, Amsterdam, vol. 29 (1926), pp. 611–641) that two polyhedra are equivalent if and only if any of their simplicial subdivisions are related by a sequence of simple transformations. Homology groups are introduced and their invariance under subdivision is proved. There is no discussion of the cohomology groups. The intersection is presented with application to duality on a manifold. The fundamental group is introduced and applied to the theory of covering complexes. The book concludes with an all too brief discussion of the author's own invention: homotopy chains.

The book is well illustrated but contains too few examples. It is clearly intended that the book is to supplement the author's earlier and more elementary book (*Einführung in die kombinatorische Topologie*, Vieweg, Braunschweig, 1932) which does contain many examples and provides the requisite knowledge of group theory. The notation is concise and well organized. This enforces slow reading.

N. E. STEENROD

## NOTES

During recent years the Department of Mathematics of the University of Chicago has planned to place emphasis on a particular field each Summer Quarter. Algebra was stressed in 1938, and analysis will be emphasized in the summer of 1939. A feature of the program will be a seminar on the calculus of variations conducted by G. A. Bliss. Other members of the summer staff who will participate in the seminar are Max Coral, L. M. Graves, M. R. Hestenes, E. J. McShane, W. T. Reid, and M. F. Smiley. In the period from Tuesday, June 27, to Friday, June 30, there will also be a Conference on the Calculus of Variations, including a number of special addresses by members of the staff and others. The speakers and their topics are as follows: *Length and area*, and *Geometrical approach to the Plateau problem*, by Tibor Radó (Ohio State); *The problem of Plateau-Riemann*, and *Minimal surfaces of higher topological structure*, by Jesse Douglas; *Existence theorems for multiple integrals*, by E. J. McShane (Virginia); *The equations of Haar and differentiability of their solutions*, by Max Coral (Wayne University); *The Jacobi condition for multiple integrals*, by L. M. Graves (Chicago); *The field theory for multiple integrals*, by G. A. Bliss (Chicago); *Sufficiency proofs by expansion methods*, by W. T. Reid (Chicago); *Functional topology and analysis in the large*, and *Variational theory in the large*, by Marston Morse (Institute for Advanced Study); *The problem of Bolza*, by M. R. Hestenes (Chicago); *Logical analysis of the semi-continuity properties of line integrals*, by Karl Menger (Notre Dame). A series of lectures preparatory to the addresses of the Conference will be given in the seminar in the preceding week, June 21–24. All mathematicians who may be interested are cordially invited to attend the Conference and the preparatory lectures in the seminar. Housing accommodations have been arranged near the University. A limited number of rooms will be available in Judson Court for the week of the Conference at the moderate charge of $2.75 per day for room and board. Judson Court is a conveniently located and modern residence hall facing the Midway Plaisance. Reservations and requests for information should be addressed to M. R. Hestenes, Eckhart Hall, University of Chicago.

The Fifth Annual Research Conference on Economics and Statistics under the auspices of the Cowles Commission for Research in Economics will be held at Colorado College, July 3–28, 1939. Modestly priced living accommodations will be provided by the College. Those interested may write to the Commission for a preliminary announcement of the program.

The Paris Academy of Sciences has announced the award of the following prizes in mathematics for 1938: Poncelet Prize, to S. Mandelbrojt, for his work in analytic functions; Francoeur Prize, to J. Dieudonné, for his work in the theory of functions; Grand Prize for Mathematical Sciences, to J. Favard, for his work in algebra and geometry; Albert I$^{er}$ de Monaco Prize, to A. Denjoy, for his mathematical work as a whole; Marquet Prize, to R. Bricard, for his work in geometry; Henri Becquerel Foundation Prize, to P. Delens, for his work in analysis and geometry; Girbal-Baral Foundation Award, to F. Marty and F. Roger.

The Lobatchewsky Prize has been awarded by the Mathematical Institute of Kazan to Professor Élie Cartan.

The Royal Society has awarded a medal to Professor R. A. Fisher for his work in mathematical statistics.

The Malaxa Prize has been awarded to Professor Jean Leray of the University of Nancy.

The Mittag-Leffler Institute Gold Medal has been awarded to Emile Picard and David Hilbert.

The Rumford Medal has been awarded by the Royal Society to Professor R. W. Wood for his work in physical optics.

Dr. F. B. Jewett, Vice President of the American Telephone and Telegraph Company and President of the Bell Telephone Laboratories, has been awarded the 1939 John Fritz Gold Medal.

Professor G. T. Whyburn of the University of Virginia has been awarded the Chauvenet Prize by the Mathematical Association of America.

The Order of Merit has been conferred by King George upon Sir James Jeans.

During his visit to Harvard University on an exchange professorship, Professor A. Denjoy of the Sorbonne gave several lectures at Princeton, Cornell, and Columbia Universities, and the Institute for Advanced Study.

Professor J. L. Coolidge of Harvard University was awarded an honorary degree in October, 1938, by Lehigh University.

Professor Solomon Lefschetz of Princeton University is delivering a set of lectures in Brussels on an appointment by the Belgian American Educational Foundation, Inc.

Professor Marston Morse of Princeton University has been elected Vice President of the A.A.A.S. and Chairman of Section A. Professor W. M. Whyburn of the University of California at Los Angeles is a member of the sectional committee.

Professor B. P. Reinsch, chairman of the department of mathematics at Florida Southern College, has been elected President of the Florida Academy of Sciences.

Professor Harlow Shapley, director of the Harvard College Observatory, has been elected a member of the Royal Swedish Academy of Sciences.

Professor G. Bouligand of the University of Poitiers has been appointed to a professorship at the Sorbonne.

Dr. Wolfgang Krull of the University of Erlangen has been appointed to a professorship at the University of Bonn.

Dr. Herbert Seifert of the University of Heidelberg has been appointed to a professorship at the University of Frankfurt.

Professor Nicola Spampinato of the University of Catania has been transferred to the University of Naples.

Assistant Professor Teresa Cohen of Pennsylvania State College has been promoted to an associate professorship.

Professor Enrico Fermi of the University of Rome has accepted a position at Columbia University beginning early in 1939.

Associate Professor G. A. Hedlund of Bryn Mawr College has been appointed to a professorship at the University of Virginia.

Dr. H. L. Krall of Pennsylvania State College has been promoted to an assistant professorship.

Professor Otto Neugebauer of Copenhagen, formerly of Göttingen, the well known mathematical historian and former editor of the Zentralblatt für Mathematik, has accepted a call from Brown University. He will take up residence in the autumn of 1939.

Associate Professor W. V. Parker of Louisiana State University has been promoted to a professorship.

Associate Professor N. E. Rutt of Louisiana State University has been promoted to a professorship.

Dr. G. E. Schweigert of the University of Virginia has accepted a position at Wright Junior College of Chicago.

Professor W. B. Ford and Associate Professor V. C. Poor, both of the University of Michigan, have sabbatical leaves for the spring term.

Professor A. W. Hobbs of the University of North Carolina is on leave of absence for the winter quarter.

Associate Professor L. T. Moore, Mr. Thomas Nicholson, and Mr. A. W. Landers, all of Brooklyn College, have sabbatical leaves for part or all of this year.

Dr. D. C. Harkin of Alabama Polytechnic Institute has an appointment as tutor at Brooklyn College for 1938–1939.

The following appointments to instructorships are announced: Brooklyn College (Evening Session): Dr. Aaron Fialkow; Rice Institute: Dr. F. E. Ulrich.

Professor Samuel Dumas of the University of Lausanne died September 24, 1938, at the age of fifty-eight.

Professor Ellen L. Burrell, head of the department of pure mathematics at Wellesley College until her retirement in 1916, died on December 3, 1938, at the age of eighty-eight years.

Dr. Fabian Franklin, professor of mathematics in the Johns Hopkins University from 1879 to 1895 and subsequently editor of the Baltimore News and associate editor of the New York Evening Post, died on January 8, 1939, at the age of eighty-six years.

Mr. P. S. Wagner of Lebanon Valley College died December 5, 1938. Mr. Wagner was a member of the Society since 1928.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

### 109. J. H. Blau: *A characterization of algebraic fields of prime characteristic.*

A field $F$ is said to be properly generated if it has a proper subring whose quotient field is $F$. It is proved that a field fails to be properly generated if and only if it is a finite field or an algebraic extension of a finite field. The general proof makes use of the well-ordering theorem, and hence does not furnish examples of such subrings in particular instances, such as the field of real numbers. (Received January 27, 1939.)

### 110. S. S. Cairns: *Planes transversal to polyhedral manifolds.*

A polyhedral manifold $P^r$ is a finite complex of euclidean simplexes in $E^n$, each point of which has an $r$-cell for a neighborhood on $P^r$. It is a Brouwer manifold if every star of simplexes has a piecewise linear homeomorph in $E^r$. Every $P^r$ with $r \leqq 3$ is a Brouwer manifold. This is not known to be true for $r > 3$. The writer investigates conditions under which one can define, through every point $p$ on $P^r$, a continuously varying transversal $(n-r)$-plane, $\pi^{n-r}(p)$. Such a plane is analogous to the normal plane to a differentiable manifold and is characterized by the existence of a neighborhood of $p$ no secant of which is parallel to $\pi^{n-r}(p)$. A necessary condition that $\pi^{n-r}(p)$ exist is that $P^r$ be a Brouwer manifold. A sufficient condition, assuming the vertices of $P^r$ independent in $E^n$, is as follows. Let $\Pi(S^k)$ be the space whose elements are the $(r-k)$-planes transversal to a star $S^k$ of simplexes on a Brouwer $k$-manifold $(k < r)$ with the vertices independent in $E^r$. Then every $(r-1-k)$-sphere in any $\Pi(S^k)$, $(k = 1, \cdots, r-1)$, must bound a singular $(r-k)$-cell in that $\Pi(S^k)$. This sufficient condition is satisfied for $r \leqq 3$. For $r = 4$, it also holds on the basis of a theorem stated by Tietze but apparently not proved in the literature. (Received January 27, 1939.)

### 111. J. E. Eaton: *Associative multiplicative systems.*

This paper is concerned with an algebraic system in which an associate operation is so defined that the product of two non-void subsets is a non-void subset. Conditions on subsets are obtained which insure that they have properties analogous to those of subgroups of groups. Of particular interest are the subsets for which a coset decomposition is possible, and study is made of the homomorphisms so generated. A class of subsets is defined which forms a Dedekind structure. Hence for this class the usual decomposition theorems are valid. All the conditions imposed on subsets in this paper are conditions on the subsets as a whole, rather than on individual members of the subsets. Application is made to the theory of multigroups and also to the theory of ordinary groups. In particular, if the multiplicative system is considered as a multi-

group, several of the theorems derived represent improvements of known theorems in the theory of multigroups. (Received January 25, 1939.)

112. J. M. Feld: *Conformal and inversive invariants of plane curves and horn angles.*

E. Kasner has found the unique absolute differential conformal invariant of the third order for horn angles of the first order, that is, curvilinear angles formed by an ordered pair of arcs which pass through a common point in a common direction and have contact of the first order at the common point (Proceedings of the Fifth International Congress of Mathematicians, vol. 2, Cambridge, 1912). Relative conformal invariants of horn angles were determined by Ostrowski (Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 44 (1934)) and independently by Kasner and Comenetz (Proceedings of the National Academy of Sciences, vol. 22 (1936)). In this paper these invariants are shown to be consequences of two formulas, one expressing the ordinary second derivative and the other the Schwarzian derivative in terms of new variables. A new expression for Kasner's absolute horn angle invariant of the third order in terms of Schwarzian derivatives is found. In addition, expressions are found for inversive invariants of curves and curvilinear angles in terms of Schwarzian derivatives and what in this paper are defined to be higher Schwarzian derivatives. (Received January 24, 1939.)

113. F. C. Gentry: *Cremona involutions determined by a pencil of surfaces.*

To each point of a line $d$ are made to correspond $k$ surfaces of a pencil of surfaces of order $n$ containing $d$ as an $(n-2)$-fold line in its base. A generic point $P$ of space determines a surface $S_n$ of the pencil. The line joining $P$ to the point of $d$ corresponding to $S_n$ meets $S_n$ in one further point $Q$, the inverse of $P$ in an involution. The effect on the order of the transformation of the basis curve of the pencil residual to $d$ becoming composite is considered, and the configuration of the $F$ curves of the second kind is determined for each of the cases which arise. (Received January 9, 1939.)

114. Philip Hartman: *An asymptotic formula for exponential integrals.*

In this paper, conditions for the asymptotic formula (1) $\int_0^1 g(x)\exp(-sx^\alpha)dx \sim Cg(+0)s^{-1/\alpha}$, $|s| = |\sigma+it| \to \infty$, are discussed. The conditions for the two regions $|\arg s| \leqq \pi/2$, $|\arg s| \leqq \pi/2-\epsilon$ are of different character. In the half-plane $|\arg s| \leqq \pi/2$, it is shown that the asymptotic formula (1) holds uniformly if $\alpha>1$ and if $g(x)$ is of bounded variation. In an angular region $|\arg s| \leqq \pi/2-\epsilon$, (1) holds uniformly for $\alpha>1$ if it is merely supposed that $g(x)$ is summable and $g(+0)$ exists; if one desires the formula for $\alpha>0$, one must require $|g(x)-g(+0)|\,|\log x|^{1/\alpha} \to 0$, $x \to 0$, in addition. The known results (for example, A. Wintner, Proceedings of the National Academy of Sciences, vol. 20 (1934), pp. 57–62) involving the order of the remainder term may be deduced with an improvement in the conditions from the results obtained here. (Received January 25, 1939.)

115. Charles Hopkins: *An extension of a theorem of Remak.*

Let $G$ be a group with minimal condition for subgroups which are admissible with respect to a set $\Omega$ of (proper) automorphisms of $G$, where $\Omega$ contains all the inner automorphisms of $G$. Then the union $M$ of all minimal admissible subgroups of $G$ is the direct product of a finite number of minimal admissible subgroups; in any two

such direct decompositions of $M$ the number of components is the same and "corresponding" components are centrally isomorphic. (Received January 23, 1939.)

116. Nathan Jacobson: *An application of E. H. Moore's determinant of a hermitian matrix.*

It is remarked that E. H. Moore's definition of a determinant holds for hermitian matrices with elements in a general type of ring. As a consequence the following theorem is proved: If $B$ is a matrix of $2n$ rows and columns with elements in a field of characteristic not equal to 2 such that $R^{-1}B'R=B$ where $R$ is any nonsingular skew-symmetric matrix, then the characteristic polynomial of $B$ has the form $[\phi(x)]^2$ and $\phi(B)=0$. (Received January 28, 1939.)

117. Mark Kac: *Statistics and power series with gaps.*

Let $\{a_k\}$ be a real bounded sequence for which $\sum a_k^2=\infty$, and $\{n_k\}$ a sequence of positive integers for which $n_k/(n_k+\cdots+n_{k-1})\to\infty$ as $k\to\infty$. Denoting by $F(z)$ the gap series $\sum a_k z^{n_k}$, ($|z|<1$), and by $[m(r)]^2$ the mean value of $|F(z)|^2$ on the circle $|z|=r$, ($r<1$), one has the following theorem: If $E$ denotes a rectangle in the complex $F$-plane and $2\pi r\mu_r(E)$ the sum of the lengths of those arcs of the circle $|z|=r$ on which $F(re^{i\phi})/m(r)$ is a point of $E$, then the distribution function $\mu_r(E)$ tends, as $r\to 1$, to a two-dimensional Gaussian distribution function which has density $\pi^{-1}\exp\{-(u^2+v^2)\}$, where $u+iv=F$. (Received January 23, 1939.)

118. Edward Kasner and J. J. DeCicco: *Curvature element transformations which preserve integrable fields.*

By a curvature element is meant simply a point of a curve together with the first two derivatives of the curve at the point. A curvature element is given by $(x, y, y', y'')$. A field consists of $\infty^3$ curvature elements. If there exist $\infty^1$ curves whose curvature elements coincide exactly with the curvature elements of a given field, then the field is termed an integrable field. In this paper, the authors find the group $G$ of all curvature element transformations which convert every integrable field into an integrable field. The fundamental result is that the group $G$ is identical with the group of extended contact transformations of lineal elements. Thus the group $G$ is identical with the group of contact transformations of curvature elements. That is, any transformation of $G$ converts every curve into a curve. The solution of the problem gives a new characteristic property of the group of extended contact transformations of lineal elements (cf. the discussion of an analogous characterization of space contact transformations in a paper appearing in the Duke Mathematical Journal). (Received January 9, 1939.)

119. E. R. Kolchin: *On the basis theorem for systems of differential polynomials.*

A finite subset $\Phi$ of a system $\Sigma$ of differential polynomials is called a basis of $\Sigma$ if for each differential polynomial $G$ in $\Sigma$ there exists a positive integer $p$ such that $G^p$ is in the differential ideal of $\Phi$. The basis is called uniform if a single $p$ can be used for all $G$'s in $\Sigma$. H. W. Raudenbush has shown (Transactions of this Society, vol. 36 (1934), pp. 361–368) that every system has a basis. In this note an example is given of a perfect differential ideal of differential polynomials which contains no uniform basis. (Received January 6, 1939.)

120. Saunders MacLane and O. F. G. Schilling: *Zero-dimensional branches of rank one on algebraic varieties.*

If $L$ is a field of algebraic functions of $n$ variables over a field of constants $K$, a zero-dimensional rank one valuation $V$ of $L$ is a real-valued function $V(a)$ defined on $L$ with the usual valuation properties, and such that $V(a)=0$ for $a$ in $K$, while the residue class field of $V$ is an algebraic extension of $K$. The value group of $V$ is the group of all real numbers $V(a)$ which occur as values. In Zariski's investigations of the resolution of the singularities of algebraic surfaces, one preliminary step is the enumeration of the possible value groups for such a $V$ over a surface ($n=2$). In this paper the authors investigate the corresponding question for $n$-dimensional varieties. It is proved that the value group either consists of all integral linear combinations of $n$ incommensurable real numbers (a group of rank $n$), or else consists of certain linear combinations with rational coefficients of $r$ real numbers, where the rank $r$ is less than $n$. Conversely, any such groups can be realized as value groups, as is shown by two constructions, one using algebraic arguments and one a Puiseux argument. (Received January 23, 1939.)

121. Harlan C. Miller: *Some methods of characterizing certain atriodic continua.*

In this paper the following results are obtained for metric spaces: I. In order that a compact continuum which contains no indecomposable subcontinuum should be an atriodic continuum, every subcontinuum of which is unicoherent, it is necessary and sufficient that every subcontinuum of it be irreducible between two of its points. II. A compact continuum every subcontinuum of which is irreducible between two of its points is atriodic but not necessarily unicoherent. III. If $M$ is a compact continuum, the following statements are equivalent: (1) $M$ is atriodic and every subcontinuum of it is unicoherent. (2) If $A$ and $B$ are two points of $M$, not more than one subcontinuum of $M$ is irreducible between them, and every subcontinuum of $M$ is irreducible between two of its points. (3) Of any three points of $M$, there is one that weakly separates the other two in $M$. (4) If $A$, $B$, and $C$ are three points of $M$, and $H$, $K$, and $L$ are three subcontinua of $M$ which are irreducible between $A$ and $B$, $A$ and $C$, and $B$ and $C$, respectively, one of the continua $H$, $K$, $L$ is the sum of the other two. (Received January 13, 1939.)

122. D. C. Murdoch and Oystein Ore: *Ideal theory in non-associative rings.*

This paper contains an axiomatic study of the properties required in a ring in order that the ordinary decomposition theorems for ideals shall hold. It is shown that they depend on the existence of certain weak associative and distributive laws. (Received January 23, 1939.)

123. Tadasi Nakayama: *On Frobeniusean algebras.* I.

Let $A$ be an algebra with a unit element over a field $F$. $A$ is called Frobeniusean if the right (first) and the left (second) regular representations are equivalent, while $A$ is called quasi-Frobeniusean if the totality of distinct directly indecomposable components of the right regular representation coincides with that of the left regular representation. The present note studies the structure of these types of algebras. The main theorem states: $A$ is Frobeniusean if and only if (i) $l(r(L))=L$, $r(l(R))=R$, and (ii) $(L:F)+(r(L):F)=(A:F)$ for every left ideal $L$ and right ideal $R$, while $A$ is

quasi-Frobeniusean if and only if (i) (but not necessarily (ii)) holds for every $L$ and $R$; where $r(S)$ $(l(S))$, $(S \subset A)$, denotes the set of right (left) annihilators of $S$ in $A$. Frobeniusean algebras have some bearings on principal ideals too. For instance, a necessary and sufficient condition that every two-sided ideal of $A$ can be expressed as $Ac = cA$ is that every residue class algebra of $A$ should be Frobeniusean. (Received January 24, 1939.)

124. Oystein Ore: *Chain properties in finite groups.*

This paper contains a study of the properties of non-normal chains of subgroups in finite groups. (Received January 25, 1939.)

125. H. J. Riblet: *Symmetric differential expressions.*

This paper is concerned with the study of the algebraic and differential properties of differential fields obtained by adjoining the roots of an algebraic equation, with coefficients in a differential field $\bar{K}$, to $\bar{K}$. The theorem which serves as a foundation for the paper is as follows. Let $y_1, y_2, \cdots, y_n$ be members of a commutative ring $R_1$ in which an operation differentiation obeying the usual sum and product rules is defined. Then any rational integral function of the $y$'s and their derivatives, with coefficients in a commutative ring $R_2$, which is unchanged by all permutations of the $y$'s is $d$ times a rational integral function $F$ of the elementary symmetric functions of the $y$'s and their derivatives. $F$ has coefficients in $R_2$ and $d$ is the reciprocal of a non-negative power of the discriminant of the $y$'s. (Received January 25, 1939.)

126. Abraham Sinkov: *A note on a paper by J. A. Todd.*

In a recent study of the linear fractional group, J. A. Todd obtained complete definitions for both $LF(2, 2^n)$ and $LF(2, p^n)$ in terms of $n+2$ generators. A further investigation of these operators shows that, in both cases, two of them, of periods two and three, suffice to generate the entire group. In terms of these two generators, the number of conditions required by Todd's definition is reduced to $n+5$ in the case of $LF(2, 2^n)$ and $2n+6$ in the case of $LF(2, p^n)$. That these restrictions may not all be independent is indicated by the examination of some special cases. Thus it results that $LF(2, 2^4)$ and $LF(2, 3^2)$ can be completely defined by means of five conditions and $LF(2, 2^5)$ by means of six. All of these latter definitions are new. In addition, the results in connection with $LF(2, 2^4)$ are amplified so as to yield necessary and sufficient conditions which must be satisfied by two operators of periods two and three if they are to generate the entire group. (Received January 26, 1939.)

127. S. M. Ulam: *On $\epsilon$-isomorphic transformations.* Preliminary report.

A transformation $T$ of a group $G_1$ into a metric group $G_2$ is called an $\epsilon$-isomorphism if $\rho(T(x_1 \cdot x_2), T(x_1) \cdot T(x_2)) \leqq \epsilon$ for all $x_1, x_2$ in $G_1$, $\rho$ denoting the metric distance in $G_2$. If $G_2$ is compact and if, for every positive $\epsilon$ there is an $\epsilon$-isomorphic transformation of $G_1$ into $G_2$, there exists a nontrivial homomorphism of $G_1$ into $G_2$. Continuous $\epsilon$-isomorphisms are studied and their behavior determined for particular groups. (Received January 28, 1939.)

128. Abraham Wald and Jacob Wolfowitz: *Confidence limits for continuous distribution functions.*

Denote by $X$ a random variable and by $f(x)$ its distribution function, that is, $f(x)$ is the probability that $X < x$. Consider two nonnegative and continuous functions

$\delta_1(y)$ and $\delta_2(y)$, defined for $0 \leqq y \leqq 1$, for which $y+\delta_1(y)$ and $y-\delta_2(y)$ are monotonically increasing in the strict sense. Denote by $\phi_n(x)$ the relative frequency of the values less than $x$ in a sample $x_1, x_2, \cdots, x_n$ of the size $n$. It is shown that the probability that $f(x)-\delta_2(f) \leqq \phi_n(x) \leqq f(x)+\delta_1(f)$ shall hold for all values $x$, depends only on $n$ and on the functions $\delta_1(y)$ and $\delta_2(y)$, provided $f(x)$ is continuous. Denote this probability by $P\{n; \delta_1(y); \delta_2(y)\}$. The formula for it has been calculated. If $f(x)$ is unknown but continuous and only a sample of the size $n$ is known, the upper and lower confidence limits of $f(x)$ corresponding to a certain probability level $\alpha$, $(0<\alpha<1)$, can be determined as follows: The functions $\delta_1(y)$ and $\delta_2(y)$ are chosen so that $P\{n; \delta_1(y); \delta_2(y)\}=\alpha$. Each of the equations $\delta_2[\phi_n(x)+\Delta_1]=\Delta_1$, $\delta_1[\phi_n(x)-\Delta_2]=\Delta_2$ has a unique solution in $\Delta_1$ and $\Delta_2$, respectively. Denote the roots by $\Delta_1(x)$ and $\Delta_2(x)$. Then the upper confidence limit is given by $\phi_n(x)+\Delta_1(x)$ and the lower confidence limit by $\phi_n(x)-\Delta_2(x)$. That is to say if, from a population whose distribution $f(x)$ is unknown but continuous, one draws a sample of size $n$, the probability is $\alpha$ that the sample will have a distribution $\phi_n(x)$ for which $\phi_n(x)-\Delta_2(x) \leqq f(x) \leqq \phi_n(x)+\Delta_1(x)$. (Received January 25, 1939.)

### 129. A. D. Wallace: *Non-boundary sets.*

This paper is to appear in volume 45 of this Bulletin. (Received January 21, 1939.)

### 130. A. D. Wallace: *Some characterizations of interior transformations.*

In this paper the author characterizes interior transformations on connected metric spaces as those continuous transformations for which the inverse of the closure is the closure of the inverse. As a consequence it follows that numerous descriptive point set theoretic properties are invariant under the inverse of an interior transformation. If in addition it is assumed that closed sets map into closed sets, it is shown that the inverse of every point is compact. With the aid of this result it is seen that every normal transformation (as defined by Stoilow) is light, that is, the inverse of every point is totally disconnected. This paper will appear in the American Journal of Mathematics. (Received January 28, 1939.)

### 131. J. L. Walsh: *On the circles of curvature of the images of circles under a conformal map.*

Under a smooth conformal map of the region $|z|<1$ onto a region $R$ of the extended plane with boundary $B$, no circle of curvature of the image of a circle $|z|=r<2-3^{1/2}$ can cut $B$. (Received January 7, 1939.)

### 132. Hassler Whitney: *On products in a complex.* II.

Duals $D(\sigma)$ to cells $\sigma$ in a complex $K$ are defined. If two systems $D$, $D'$ are in general position, then in each $\sigma^{r+s}$, each pair $D(\sigma^r)$ and $D'(\sigma^s)$ has a Kronecker index $\{D(\sigma^r)\cdot D'(\sigma^s)\}_{\sigma^{r+s}}$. These are shown to satisfy the conditions in a previous paper (Annals of Mathematics, (2), vol. 39 (1938), p. 403). Special systems $D$, $D'$ define the products of §6, loc. cit. "Locally equivalent complexes" are defined, and the products in them considered, which generalizes work of de Rham (Commentarii Mathematici Helvetici, vol. 4, p. 151); in addition, the product of two 1-chains, coefficients modulo 2, can give a 2-chain with integral coefficients. Some special topics are also considered. This paper is necessary in the general theory of sphere-spaces. (Received January 26, 1939.)

133. Norbert Wiener and Brockway McMillan: *A new method in statistical mechanics.*

The calculus of the discrete homogeneous chaos is developed and extended to permit applications in the field of statistical mechanics. In particular, as a model for other applications, a discrete homogeneous chaos $M(P;\alpha)$, representing a distribution of phase points in a six-dimensional phase space, is used as initial conditions for the dynamical equations of an infinite monatomic gas with intermolecular forces. The transformation in time of the chaos in accordance with the laws of dynamics enables one to calculate the time derivatives of linear, quadratic, and higher degree functionals of the chaos. These are at time $t=0$ simply quadratic and higher degree functionals of $M(P;\alpha)$, and this permits a large class of statistical parameters of the gas to be expanded, at least formally, in power series in the time. The ergodic theorem enables one to calculate these statistical parameters for "almost all" dynamical systems in the ensemble in question by a simple averaging over the $\alpha$ of the initial chaos. Methods of extending these statistical parameters to their equilibrium values at $t=\infty$ will then give information about the physical gas. This extension is made for the mechanical analogue of temperature. (Received January 26, 1939.)

134. Alonzo Church: *On the concept of a random sequence.*

The following modification of von Mises' definition of "random sequence" (*regellose Folge*) is proposed. An infinite sequence $a_1, a_2, \cdots$, of 0's and 1's, for example, is a random sequence (1) if the relative frequency of 1's among the first $r$ terms of the sequence approaches a limit $p$ as $r\to\infty$; and (2) if $\phi$ is any effectively calculable function of positive integers in the sense of the author, (American Journal of Mathematics, vol. 58, pp. 345–363), if $b_1=1$, $b_{n+1}=2b_n+a_n$, $c_n=\phi(b_n)$, and the integers $n$ such that $c_n=1$ form in order of magnitude an infinite sequence $n_1, n_2, \cdots$, then the relative frequency of 1's among the first $r$ terms of $a_{n_1}, a_{n_2}, \cdots$ approaches $p$ as $r\to\infty$. The validity of this rests on the idea that a *Spielsystem* (von Mises, *Wahrscheinlichkeit, Statistik und Wahrheit*) should be thought of, not as a function, but as an effective algorithm for the calculation of the values of a function. Since the set of effectively calculable functions is (non-effectively) enumerable, it seems likely that a (non-constructive) existence proof for random sequences in the present sense might result from an extension of the methods of Copeland (American Journal of Mathematics, vol. 58, pp. 181–192). (Received February 2, 1939.)

135. Douglas Derry: *On finite abelian p-groups.* I.

Let $S_1, \cdots, S_n$ be a set of elements of a finite abelian $p$-group $\mathfrak{G}$ whose composition series has length $n+1$ such that, for any subgroup $\mathfrak{H}$ of $\mathfrak{G}$ with cyclic factor group $\mathfrak{G}/\mathfrak{H}$, elements $S_{m_1}, \cdots, S_{m_k}$ exist so that the group series $(S_{m_1}, \cdots, S_{m_k}, \mathfrak{H}), \cdots, (S_{m_1}, \mathfrak{H})$, $\mathfrak{H}$ is a composition series from $\mathfrak{G}$ to $\mathfrak{H}$. The present paper shows that if the rank of $\mathfrak{G}$ be 2, the elements $S_1, \cdots, S_n$ after a possible rearrangement of order build a composition series $(S_1, \cdots, S_n), (S_1, \cdots, S_{n-1}), \cdots, (S_1), (E)$ for $\mathfrak{G}$. (Received January 31, 1939.)

136. R. J. Duffin and A. C. Schaeffer: *A problem in metric diophantine approximation.*

By means of continued fractions, Khintchine has shown that if a sequence of positive numbers satisfies the conditions $\sum a_n=\infty$ and $na_n$ decreases monotonically, then for almost every real number $x$ there are arbitrarily large integers $n$ and $m$ for which $|nx-m|<a_n$. We have found a method of proving this theorem without the use of

continued fractions. Moreover, we replace the condition $na_n$ monotonically decreasing by the less stringent condition $a_n$ monotonically decreasing. Our most general result may be put in this form: let $\{b_i\}$ be any sequence of positive numbers for which $\sum b_i = \infty$, and let $\{n_i\}$ be an infinite sequence of integers for which $\phi(n_i)/n_i$ has a positive lower bound, where $\phi$ is Euler's $\phi$-function; then for almost every real number $x$ there are arbitrarily large integers $i$ and $m$ for which $|n_i x - m| < b_i$. (Received February 13, 1939.)

137. H. T. Engstrom: *A note on fundamental systems of symmetric functions.*

The author proves that any $n$ polynomials, symmetric in $n$ variables, over a field of characteristic zero form a fundamental system if they are independent and the product of their degrees is less than $2n!$. The proof is based on Perron's theorem concerning the degree of the dependence relation between $n+1$ polynomials in $n$ variables. (Received February 2, 1939.)

138. L. M. Graves: *The Weierstrass condition for multiple integral problems of the calculus of variations.*

The Weierstrass $\mathcal{E}$-function for the multiple integral $I = \int f(t, x, p)dt$ has the form $\mathcal{E}(t, x, p, P) = f(t, x, P) - f(t, x, p) - (P_{\alpha i} - p_{\alpha i}) f_{\alpha i}(t, x, p)$, where $t = (t_1, \cdots, t_n)$, $x = (x_1, \cdots, x_m)$, $p = (p_{\alpha i}) = (\partial x_i / \partial t_\alpha)$, $f_{\alpha i} = \partial f / \partial p_{\alpha i}$, $i = 1, \cdots, m$, $\alpha = 1, \cdots, n$. A necessary condition for a minimum of $I$ is that $\mathcal{E} \geq 0$ along the minimizing manifold whenever the matrix $(P_{\alpha i} - p_{\alpha i})$ has rank one. This has been proved by McShane under very general hypotheses (see Annals of Mathematics, vol. 32 (1931), pp. 578–590). The present paper gives an extremely brief and elementary proof, applicable when the minimizing manifold $x_i = x_i(t)$ has continuous first derivatives $p_{\alpha i}$. For problems in parametric form the integrand $f$ is independent of the variables $t_\alpha$, and equals a function $g(x, j)$ of $x_i$ and the $n$-rowed minors $j_\rho$ of the matrix $(p_{\alpha i})$. When these minors $j_\rho$ are arranged as an $m$-rowed matrix $k = (k_{it})$ in a certain way, it is seen that the rank of $k$ is always $(m-n)$. The matrix $(P_{\alpha i} - p_{\alpha i})$ has rank one if and only if the matrix $(k_{it}, K_{iu})$ has rank $(m-n+1)$. It is necessary for a minimum of $I$ that $\mathcal{E}(x, j, J) = g(x, J) - g(x, j) - (J_\rho - j_\rho) g_\rho \geq 0$ for all $J_\rho$ such that this condition is satisfied. This result was proved directly for parametric problems for the cases $n = 1$ and $n = m - 1$ by McShane (ibid., pp. 723–733). (Received January 30, 1939.)

139. Nathan Jacobson: *The fundamental theorem of the Galois theory for quasi-fields.*

Let P be an arbitrary quasi-field and $\mathfrak{G}$ a finite group of $n$ outer automorphisms in P. If $\Phi$ is the set of invariant elements, either order $(P:\Phi) = n$ and there is a (1–1) correspondence between the subgroups of $\mathfrak{G}$ and the sub-quasi-fields $\Sigma$ between $\Phi$ and P. The proof of this extension of the fundamental theorem of Galois theory is obtained by considering the ring of operators $\sum S\xi_S$ acting in P where $S \in \mathfrak{G}$ and $\xi$ is the operator $\alpha \to \alpha\xi$. The theorem has applications to division algebras. The method used gives also the following extension of a well known theorem of Hilbert: Let $U_S$ be a set of matrices corresponding to the $S$'s of $\mathfrak{G}$ such that $U_1 = 1$ and $U_T U_S^T = U_{ST}$, then there exists a matrix $A$ such that $U_S = A^{-1} A^S$. (Received February 14, 1939.)

140. D. H. Lehmer: *On the remainders and convergence of the series for the partition function.*

The two series under discussion are the asymptotic series of Hardy and Ramanujan (1917) and the convergent series of Rademacher (1937) for the number $p(n)$ of

unrestricted partitions of $n$. (See, for instance, Transactions of this Society, vol. 43 (1938), pp. 271–295.) The writer has given estimates for the remainders $R_1(n, N)$, $R_2(n, N)$ after $N$ terms of these respective series of such a form that in either case $R_i(n, \alpha n^{1/2}) = O(n^{-1/2})$, as compared with the previous result of $O(n^{-1/4})$. In the present paper the author obtains more precise estimates for which $R_i(n, \alpha n^{1/2}) = O(n^{-1/2} \log n)$, a result which is, in a sense, the best possible. The divergence of the Hardy-Ramanujan series was proved by the writer in 1937 by a method which did not remove the possibility of the series oscillating between fixed limits. In this paper the author gives a new proof of divergence showing that the terms of the series are in fact unbounded. For the Rademacher series it is proved that the $k$th term exceeds $1/k^2$ in absolute value for infinitely many $k$'s. A brief discussion is given of the function $p(-n)$ defined by the Rademacher series and conjectured to be identically zero. It is shown that $p(-n) = O(n^{-1/2} \log n)$ and that the remainder of its series after $N$ terms is $O(N^{-1} \log N)$, for every $n$. (Received February 4, 1939.)

141. Norman Levinson: *An inequality in function theory.*

Let $M(x)$ be an even function of $x$, monotone decreasing for positive $x$. Let $F(z)$, $z = x+iy$, be analytic in the square $(|y| < 1, |x| < 1)$, and let $|F(x+iy)|$ be less than $M(x)$ in this square. A necessary and sufficient condition that there exist a constant $C$ depending only on $M(x)$ and $\delta > 0$ such that $|F(x+iy)| < C$, $(|x| < 1-\delta, |y| < 1-\delta)$, is that $\int_0^1 \log \log M(x) dx < \infty$. (Received February 15, 1939.)

142. R. E. O'Connor: *Quadratic and linear congruence.*

The number $N(m)$ of simultaneous solutions of the integral congruences, $\sum a_{ij} x_i x_j \equiv r$, $\sum b_i x_i \equiv s \pmod m$, $(i, j = 1, 2, \cdots, n)$, is given for $m$ odd, prime to the g.c.d. of each set of coefficients, and either square-free or at least containing no prime factors of a certain class to a degree higher than the first. An application is made to 3-dimensional vectors defined with respect to a quadratic form. (Received February 13, 1939.)

143. L. F. Ollmann: *On joining finite subsets of a planar Peano space by simple closed curves.*

Let $K$ be a finite subset of a Peano space $M$ in the euclidean plane such that no two points of $K$ may be separated by the omission of any two points of $M$. Then: (1) If $K$ consists of no more than 5 points, there is always a simple closed curve $J$ such that $M \supset J \supset K$. (2) Two examples are given to show that (1) is not true for 6 points. (In one example, each point of $M$ is of Menger order $c$, while in the other each point is of order less than or equal to 4.) (3) For any $n$ a set $M$ is constructed containing $2n$ points which cannot be joined by $n-2$ mutually exclusive simple closed curves in $M$. (4) For any $n$ a cubic graph $M$ is constructed in which the vertices cannot be joined by $n$ simple closed curves (not necessarily mutually exclusive). (5) An example is given of a set $M$, which is a triangular graph (all faces are triangles), such that the vertices cannot be joined by a simple closed curve. (Received February 16, 1939.)

144. J. F. Ritt: *On ideals of differential polynomials.*

Let $\Sigma$ be an ideal of differential polynomials. Let the manifold of $\Sigma$ be separable into submanifolds none of which has a solution in common with any other. It is proved that $\Sigma$ can be represented in one and in only one way as the intersection of ideals whose manifolds are the various submanifolds. (Received February 2, 1939.)

145. A. C. Schaeffer: *The Fourier-Stieltjes coefficients of a function of bounded variation.*

Let $F(x)$, defined in the interval $(-\pi, \pi)$, be a continuous function which is of bounded variation but is not absolutely continuous, and let $c_n(F)$ denote its Fourier-Stieltjes coefficients. In a recent paper (American Journal of Mathematics, vol. 60 (1938), pp. 513–522) Wiener and Wintner, improving an earlier result of Littlewood, have shown that, if $\gamma$ is any positive number less than $1/2$, there exists an $F(x)$ such that $c_n(F) = O(n^{-\gamma})$. In this paper the method of Wiener and Wintner is used to show that if $r(n)$ is any function of $n$ which tends to infinity as $n$ tends to infinity, then there exists an $F(x)$ such that $c_n(F) = O(r(n)n^{-1/2})$. This function satisfies the additional condition, as have earlier examples, that its derivative vanishes almost everywhere. (Received February 11, 1939.)

146. W. E. Sewell: *Integral approximation and continuity in the real domain.*

In this paper the author studies the relation between the degree of approximation to a periodic function $f(\theta)$ by trigonometric sums in the sense of least $p$th powers and the continuity properties of $f(\theta)$. (Received February 10, 1939.)

147. W. E. Sewell: *Integral approximation and cortinuity in the complex domain.*

In this paper the author studies the relation between degree of approximation by polynomials in the sense of least $p$th powers measured by a line integral taken over a rectifiable Jordan curve $C$ and the continuity properties of the function approximated on the curve $C_R$ in the complex domain. (Received February 10, 1939.)

148. H. C. Trimble: *On the ring of matrices commutative with a given matrix.*

Let $R(A)$ be the ring of matrices commutative with a given $n$th order square matrix $A$ with elements in a commutative field $K$. Two isomorphisms of a certain class of isomorphisms (introduced by M. H. Ingraham and the author and similar to an isomorphism given by P. L. Trump) are established. In either case, $R(A)$ is shown to be isomorphic to a system of classes of matrices of lower order whose elements are polynomials in an indeterminate $\lambda$, and whose rows are reduced modulo certain factors of the minimal polynomial of $A$. By using these isomorphisms, $R(A)$ is determined rationally, well known properties of $R(A)$ are proved simply, and new results are obtained. The bulk of the paper is an outline of a rational attack upon the similarity problem in $R(A)$. The methods are believed suggestive, and though the theory is incomplete in one place, no example has been found to escape the methods used. The case in which $A$ has at most two invariant factors is solved completely. (Received January 26, 1939.)

149. E. P. Vance: *Generalizations of non-alternating and non-separating transformations.*

G. T. Whyburn (American Journal of Mathematics, vol. 56 (1934), pp. 294–302) and J. F. Wardwell (Duke Journal of Mathematics, vol. 2 (1936), pp. 745–750) have considered non-alternating and non-separating transformations in recent papers. It is possible to obtain interesting results by weakening the conditions of these transforma-

tions and also of completely non-alternating and completely non-separating transformations. A continuous transformation will be defined to be weakly non-alternating provided $T^{-1}(x)$ does not separate $T^{-1}(y)$ between any two points $u$ and $v$ of $T^{-1}(y)$ unless a single point does. Similar definitions can be made for the three other transformations. In this paper these four transformations are characterized, factor and product theorems are proved, and the transformations are applied to locally connected continua and to special curves and surfaces. (Received February 16, 1939.)

150. Abraham Wald: *Lower and upper limits of a distribution function determined by moments and inequalities satisfied by moments.*

Denote by $X$ a random variable and by $P(\alpha<X<\beta)$ the probability that $\alpha<X<\beta$. For any positive integer $r$ the expected value $E|X-x_0|^r$ of $|X-x_0|^r$ is called the absolute moment of the order $r$ about $x_0$. Denote by $i_1, i_2, \cdots, i_j$ $j$ different positive integers and by $M_{i_1}, \cdots, M_{i_j}$ the absolute moments of the order $i_1, \cdots, i_j$ of a certain random variable $X$ about $x_0$. For any $d>0$ the greatest lower bound $a_d$ and the least upper bound $b_d$ of the probabilities $P(-d<Y-x_0<d)$, formed for all random variables $Y$ for which $E|Y-x_0|^{i_\nu}=E|X-x_0|^{i_\nu}=M_{i_\nu}$ $(\nu=1, \cdots, j)$, are calculated. This is a generalization of the Tshebysheff-Markoff inequalities which give the upper and lower limits in question only if $i_1=1, \cdots, i_j=j$ and only for certain special values of $d$. In this paper is given also the complete system of inequalities satisfied by absolute moments. That is to say, for any $j$ different positive integers $i_1, \cdots, i_j$ the necessary and sufficient conditions are given which must be satisfied by $j$ numbers $c_1, \cdots, c_j$ in order that a random variable $X$ should exist of which the $i_\nu$th absolute moment about $x_0$ is equal to $c_\nu$, $(\nu=1, \cdots, j)$. (Received February 2, 1939.)

151. Max Zorn: *Continuous groups and Schwarz' lemma.*

This paper contains a discussions of Schwarz' lemma with elementary topological methods. The results are of use for the topological characterization of the set of all conformal mappings within a Riemann surface. (Received February 7, 1939.)

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